Riemannian geometry: a note for reviewing
2024 autumn

This is a re-arranged note for the course on Riemannian geometry given by professor Yang, which aims for a quick reviewing of the basic computations and the main results. The gist lies in the exercises. Some good references are [Pet06, Jos08, DCFF92, Wal09]. Many related topics are to be appended in the future.

1. Basic concepts and computations

1.1. Connections and curvatures

Definition 1 (connection)**.** *$\nabla : TM \times E \to E$, which is linear on TM, a derivation for E, where $E \to M$ is a bundle.*

Definition 2 (Christoffel symbol)**.** $\nabla_{\frac{\partial}{\partial x^i}} e_A = \Gamma^B_{iA} e_B$.

Definition 3 (curvature tensor)**.** *$R : TM \otimes TM \otimes E \otimes \to E$,*

$$R(X, Y)e := \nabla_X \nabla_Y e - \nabla_Y \nabla_X e - \nabla_{[X,Y]} e$$

As for a Riemannian manifold (M, g), we consider usually Levi-Civita connection, and several special curvature tensors.

Definition 4 (Levi-Civita connection)**.** *$\nabla : TM \times TM \to TM$, a connection s.t.*

(1) $X(Y, Z) = (X\nabla_Y, Z) + (Y, \nabla_X Z)$;

(2) $\nabla_X Y - \nabla_Y X = [X, Y]$.

Definition 5 (curvature tensors and operator)**.**

(1) $R(X, Y, Z, W) := (R(X, Y)Z, W)$, $R = R_{ijkl} \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \mathrm{d}x^l$;

(2) sectional curvature: $K_\sigma (= \sec(X, Y)) = \frac{R(X,Y,Y,X)}{|X \wedge Y|^2}$, $\sigma = \mathrm{span}\{X, Y\}$;

(3) Ricci curvature: $\mathrm{Ric}_{ij} = g^{kl} R_{iklj}$;

(4) Scalar curvature: $S = g^{ij} \mathrm{Ric}_{ij}$.

(5) curvature operator: $\mathfrak{R} : \wedge^2 TM \to \wedge^2 TM$, *such that* $g(\mathfrak{R}(X \wedge Y), Z \wedge W) = R(X, Y, Z, W)$.

List of properties:

- symmetry of R and first Bianchi;
- independence of basis for K_σ;
- independence of planes for K_σ iff being flat;
- for 3-dim manifolds, CRC implies CSC.

Definition 6 (trace definition of Ricci)**.** $\mathrm{Ric}(v, w) = \mathrm{tr}(x \mapsto R(x, v)w)$. *Taking an ONB of TM,*

(1) $\mathrm{Ric}(v) := \sum R(v, e_i)e_i$;

(2) $\mathrm{Ric}(v,w) = g(\mathrm{Ric}(v), w)$*;*

(3) for $v = e_1$*,* $\mathrm{Ric}(v,v) = \sum R(v, e_i, e_i, v) = \sum_{i=2}^{n} \sec(v, e_i)$*.*

Exercise 7. *(1) show the Koszul formula;*

(2) calculate Γ_{ij}^k, R_{ijkl}*;*

(3) show that $R_{ijkl} =$

$$\frac{1}{2}\left(\frac{\partial^2 g_{jl}}{\partial x^i \partial x^k} + \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{il}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l}\right) + g_{pq}(\Gamma_{ik}^p \Gamma_{jl}^q - \Gamma_{il}^q \Gamma_{kj}^p).$$

(4) compute the curvatures of S^n, H^2*;*

(5) compute the curvatures of

$$g_{ij} = \delta_{ij} + \frac{x^i x^j}{K^2 - \sum (x^i)^2}, K^2 - \sum (x^i)^2 > 0;$$

(6) compute the curvatures of $(\mathbb{R}^2, e^{a(x^2+y^2)}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y))$*.*

Exercise 8. *(1) what's the relation of curvatures between* g *and* $k \cdot g$*;*

(2) prove the integral formulae for Ric *and* S*:*

(a) for unit vector v*, and* $S_v^{\perp}$ *the set of unit vectors orthogonal to* v*,*

$$\mathrm{Ric}_p(v,v) = \frac{n-1}{\mathrm{Vol}(S^{n-2})} \int_{w \in S_v^{\perp}} \sec(v,w) \, \mathrm{d}V_{\widehat{g}} \, .$$

(b) for $UT_pM \cong S^{n-1}$*,*

$$S_p = \frac{n}{\omega_{n-1}} \int_{S^{n-1}} \mathrm{Ric}_p(v,v) \, \mathrm{d}S \, .$$

(3) let (M^3, g) *be Einstein, show that* (M, g) *is of CSC.*

(4) (hard, warped product) consider (N^{n-1}, g_N)*,* $\mathrm{Ric} = \frac{n-2}{n-1}\lambda g_N, \lambda < 0$*, find a function* $\rho : \mathbb{R} \to (0, \infty)$*, such that* $(M^n, g) = (\mathbb{R} \times N, \mathrm{d}r^2 + \rho^2 g_N)$ *becomes an Einstein metric with* $\mathrm{Ric} = \lambda g$*.*

1.2. Hessian and scalar Laplacian

Consider smooth function $f : (M, g) \to \mathbb{R}$.

Definition 9 (Hessian and saclar Laplacian)**.**

(1) $\operatorname{Hess} f := \nabla^2 f = \nabla \mathrm{d} f$, *i.e.*

$$\operatorname{Hess} f(X,Y) = g(\nabla_X \nabla f, Y) = (\nabla_X \,\mathrm{d} f) = XYf - \nabla_X Y f.$$

the Hessian operator is given by $\operatorname{Hess} f(X,Y) = (\mathcal{H}_f(X), Y)$.

(2) $\Delta_g f := \operatorname{tr} \operatorname{Hess} f = g^{ij} \operatorname{Hess} f_{ij}$.

Locally, $\operatorname{Hess} f_{ij} = \operatorname{Hess} f_{ji}$, thus $\operatorname{Hess} f$ is a symmetric 2-form.

Theorem 10 (volume expression of the Laplacian)**.**

$$\Delta_g f = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det g} \frac{\partial f}{\partial x^j} \right)$$

Exercise 11. *(1) for* $\mathrm{d}\,\mathrm{Vol}_g = \sqrt{\det g}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, *compute* $\frac{\partial \det g}{\partial x^i}$, $\frac{\partial \log \det g}{\partial x^i}$ *and* $\frac{\partial \sqrt{\det g}}{\partial x^i}$, *show*

$$\frac{\partial}{\partial x^i} \mathrm{d}\,\mathrm{Vol}_g = \frac{1}{2} \frac{\partial \log \det g}{\partial x^j} \mathrm{d}\,\mathrm{Vol}_g\,.$$

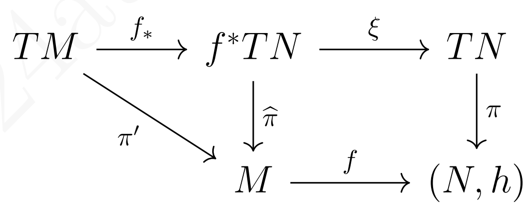
(2) prove Theorem 10.

(3) show that

$$\operatorname{Hess} \varphi(f) = \varphi'' \,\mathrm{d} f^2 + \varphi' \operatorname{Hess} f.$$

1.3. Pull-back operation

$f : M \to N$ induces $f_* : TM \to f^*TN$, for immersion, $f^*TN \subset TN$.



Theorem 12 (definition of pull-back connection and metric)**.** *There exists compatible pull-back connection and metric defined by*

(1) $\widehat{\nabla}_{\frac{\partial}{\partial x^i}} \widehat{e}_A = f_* \left(\frac{\partial f^\alpha}{\partial x^i} \nabla_{\frac{\partial}{\partial y^\alpha}} e_A \right) = f_* \left(\frac{\partial f^\alpha}{\partial x^i} \Gamma^B_{\alpha A}(f) e_B \right)$;

(2) $\widehat{g} = f^* h$, *i.e.* $\widehat{g}(\widehat{e}_A, \widehat{e}_B) = h(e_A, e_B)$.

Locally, drop the hats,

$$\widehat{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^j} = \frac{\partial f^\alpha}{\partial x^i} \Gamma^k_{j\alpha}(f) \frac{\partial}{\partial y^k};$$

$$\widehat{g}_{ij} = h\left(f_* \frac{\partial}{\partial x^i}, f_* \frac{\partial}{\partial x^j} \right) = \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} h_{\alpha\beta}.$$

Exercise 13. *(1) show the well-defined-ness and compatibility.*

(2) show that $\widehat{R}_{ij\gamma\delta} = \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j} R_{\alpha\beta\gamma\delta}$.

1.4. The 2nd fundamental form

The 2nd fundamental form, which generalize the Hessian, is defined to indicate the deviation under pull-back.

General case

Definition 14 (2nd fundamental form)**.** $B \in \Gamma(M, T^*M \otimes T^*M \otimes f^*TN)$, $B(X,Y) := \widehat{\nabla}_X f_* Y - f_* \nabla_X Y$.

Locally, $B^\alpha_{ij} = B^\alpha_{ji}$, thus B is a symmetric (2,1)-tensor, as a result,

$$\widehat{\nabla}_X f_* Y - \widehat{\nabla}_Y f_* X = f_* \nabla_X Y - f_* \nabla_Y X = f_*[X,Y].$$

Exercise 15. *(1) compute the local expression of* B.

(2) $f : (M, g) \to (N, h)$, *and* $\widetilde{\nabla}$ *is the affine connection on* $T^*M \otimes f^*TN$ *induced by* ∇^M, ∇^N, *then* $B = \widetilde{\nabla} \mathrm{d}f$, *where* $\mathrm{d}f$ *is regarded as a smooth section in* $\Gamma(M, T^*M \otimes f^*TN)$.

The case of Riemannian immersion

Given an immersion $f : M \to (\overline{M}, \overline{g}, \overline{\nabla})$, $f^*T\overline{M} \subset T\overline{M} = f^*T\overline{M} \oplus T^\perp M$. We write $(\widehat{g}, \widehat{\nabla}), (g, \nabla)$ for the induced structures on f^*TN, TM.

List of properties:

- $g_{ij} = \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\overline{g}_{\alpha\beta}$;
- $B \in \Gamma(M, T^*M \otimes T^*M \otimes T^\perp M)$, i.e. $\widehat{g}(B(X,Y), f_*Z) = 0$ for any $X, Y, Y \in \Gamma(M, TM)$. Equivalently (*drop of push-forward*),
$$\widehat{g}(\widehat{\nabla}_X f_* Y, f_* Z) = \widehat{g}(f_* \nabla_X Y, f_* Z) = g(\nabla_X Y, Z).$$
- (Gauss-Codazzi) for any $X, Y, Z, W \in \Gamma(M, TM)$,
$R(X,Y,Z,W) - \overline{R}(X,Y,f_*Z,f_*W)$
$$= \widehat{g}(B(X,W), B(Y,Z)) - \widehat{g}(B(X,Z), B(Y,W)).$$

Definition 16 (Weingarten map)**.** $X, Y \in \Gamma(M, TM), \eta \in \Gamma(M, T^\perp M)$, $g(W_\eta(X), Y) := B_\eta(X,Y) := g(B(X,Y), \eta)$.

Remark 17. *Take* $(\widehat{M}, \widehat{g}) = (\mathbb{R}^N, g_{\mathbb{R}^N})$, *we shall get Gauss' Theorema Egregium, especially for the immersion of a surface into* $\mathbb{R}^3$.

Exercise 18. *(1) show the orthogonal relation with(out) the rank theorem.*

(2) consider immersion of a surface into $\mathbb{R}^3$*, with unit normal vector* n*, write the expression of first and second fundamental form,* B_n*, and Gauss' Theorema Egregium:*

$$K = \frac{\det II}{\det I} = \sec(X,Y) = \frac{R(X,Y,Y,X)}{g_D(X,X)g_D(Y,Y) - g_D(X,Y)^2}.$$

(3) show that $\operatorname{Ric} g_D = K g_D, S = 2K$*.*

(4) consider $S^n \to \mathbb{R}^{n+1}$ *and the local parametrization*

$$\gamma : D \to U^+_{n+1} \subset \mathbb{R}^{n+1}, \gamma(u) = \left(u^1, \cdots, u^n, \sqrt{1-|u|^2}\right)$$

where $D = \{u \mid |u| < 1\}$*.*

(a) compute $g_D = \gamma^* g_{\mathrm{can}}$*;*

(b) compute the second fundamental form;

(c) compute the mean curvature $H = \frac{1}{n}\operatorname{tr}_{g_D} B$*.*

Exercise 19. *let* (M,g) *be a complete riemannian manifold. suppose* $f : M \to \mathbb{R}$ *is a smooth function with*

$$|\nabla f| = 1, \quad \operatorname{Hess} f = 0.$$

set $N = f^{-1}(0), h = g|_N$*, show that* (N,h) *is a totally geodesic submanifold of* (M,g)*.*

1.5. Parallel transports, geodesics and exponential maps

Parallel transport

Let $\gamma : I \to (M,g)$ be a smooth curve.

Proposition 20 (definition of parallel transport)**.** *For any* $v \in T_{\gamma(t_0)}M$*, there exists a unique vector field* $V \in \Gamma(I, \gamma^*TM)$ *(along* γ*) with*

(1) $V(t_0) = v$*;* *(2)* $\widehat{\nabla} V = 0$*.*

Define the parallel transport along γ *by* $P_{t_0,t,\gamma} = V(t)$*, for any* $t_0, t \in I$*.*

List of properties: the gist is a take a *parallel frame.*

- $P_{t_2,t_3,\gamma} \circ P_{t_1,t_2,\gamma} = P_{t_1,t_3,\gamma}, P_{t,t,\gamma} = \mathrm{id}$.

- $P_{s,t,\gamma} : T_{\gamma(s)}M \to T_{\gamma(t)}M$ is a linear isometry for any $s, t \in I$;
- $F(t,(s,v)) := (t, P_{s,t,\gamma}(v))$ is a smooth function;
- $\frac{\mathrm{d}}{\mathrm{d}t} P_{t,t_0,\gamma}(V(t)) = P_{t,t_0,\gamma}(\widehat{\nabla} V(t))$, for any vector field V along γ.

Exercise 21. *prove the properties above.*

GEODESIC AND EXPONENTIAL MAP

Proposition 22 (definition of geodesic). *For any $p \in M, v \in T_pM, t_0 \in \mathbb{R}$, there is an open interval $I \ni t_0$ and a smooth curve $\gamma : I \to M$ with*

(1) $\gamma(t_0) = p, \gamma'(t_0) := (\gamma_* \frac{\mathrm{d}}{\mathrm{d}t})|_{t_0} = v$;

(2) $\widehat{\nabla}\gamma' = 0$ along I.

The curve satisfying (2), i.e.

$$\widehat{\nabla}\gamma' = \widehat{\nabla}\gamma_* \frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}^2\gamma^i}{\mathrm{d}t^2}\frac{\partial}{\partial x^i} + \frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\Gamma_{ij}^k(\gamma)\frac{\partial}{\partial x^k} = 0,$$

is called a geodesic along I. Up to a shift of position, we suppose $\gamma(0) = p, \gamma'(0) = v$ and write $I_{p,v}$ for the maximal existence interval of γ.

List of properties:

- $|\gamma'|$ is a constant for the geodesic γ;
- $\gamma_{cv}(t) = \gamma_v(ct)$, i.e. invariant under rescaling.
- $P_{0,t,\gamma_v}(v) = \gamma_v'(t)$.

Definition 23 (exponential map). *Write $\mathcal{E}_p = \{v \mid 1 \in I_{p,v}\}$, the exponential map $\exp_p : \mathcal{E}_p \to M$ is defined by*

$$\exp_p(v) = \gamma_v(1),$$

where γ_v is the geodesic with $\gamma(0) = p, \gamma'(0) = v$.

List of properties:

- $\exp_p(tv) = \gamma_v(t)$, for $t \in I_{p,v}$;
- exp is smooth on $\mathcal{E} = \{(p,v) \mid v \in \mathcal{E}_p\}$;
- exp is a local diffeomorphism, since the differential

  $$\exp_{*,0} : T_0(T_pM) \to T_pM$$

  is the identity map.

- set $B_r(p) = \{\exp_p(v) \mid |v| < r\}$, then $\exp|_{B_r(p)}$ is a diffeomorphism. The injectivity radius of p is

$$\mathrm{inj}_p(M) := \sup\{r \mid \exp|_{B_r(p)} \text{ is diffeomorphic}\},$$

and $\mathrm{inj}(M) := \inf_p \mathrm{inj}_p(M)$.

Exercise 24. *prove the following Gauss' lemma: fix $p \in M, r < \mathrm{inj}_p(M)$ and I an open interval. suppose*

(1) $w(s) : I \to T_pM$ satisfies $|w(s)| = r$ and

(2) $\alpha(t,s) := \exp_p(tw(s))$ for $(t,s) \in \mathbb{R} \times I, tw(s) \in \mathcal{E}_p$.

then

$$\left\langle \alpha_* \frac{\partial}{\partial s}, \alpha_* \frac{\partial}{\partial t} \right\rangle = 0.$$

Exercise 25. *(1) let M be a smooth manifold and ∇ any connection on TM. We define the curvature endomorphism by*

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

then ∇ is said to be flat if $R(X,Y)Z \equiv 0$. show that the followings are euqivalent.

(a) ∇ is flat;

(b) for every point $p \in M$, there exists a parallel local frame defined on a neighborhood of p;

(c) for all $p, q \in M$, parallel transport along an admissible curve segment from p to q depends only on the path-homotopy class.

(d) parallel transport around any sufficiently small closed curve is the identity, i.e. for every $p \in M$, there exists a neighborhood U of p such that if $\gamma : [a,b] \to U$ is an admissible curve in U starting and ending at p, then $P_{ab} : T_pM \to T_pM$ is the identity map.

(2) a vector field X is said to be parallel if $\nabla X \equiv 0$.

(a) let $p \in \mathbb{R}^n, v \in T_p\mathbb{R}^n$, show that there is a unique parallel vector field Y on $\mathbb{R}^n$ such that $Y_p = v$.

(b) let $X(\varphi, \theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ be the spherical coordinate of an open subset $U \subset S^2$, let $X_\varphi = X_* \frac{\partial}{\partial \varphi}, X_\theta = X_* \frac{\partial}{\partial \theta}$. compute $\nabla_{X_\theta} X_\varphi, \nabla_{X_\varphi} X_\varphi$, and conclude that X_φ is parallel along the equator and along each meridian $\theta = \theta_0$.

(c) let $p = (1, 0, 0) \in S^2$*, show that there is no parallel vector field* W *on any neighborhood of* p *in* S^2 *such that* $W_p = X_\varphi|_p$*.*

(d) conclude that no neighborhood of p *in* (S^2, g) *is isometric to an open subset of* $(\mathbb{R}^2, g_{\text{can}})$*.*

1.6. Completeness

Completeness of manifolds and vector fields

A riemannian manifold is naturally a metric space under

$$d_g(p, q) = \inf_{\gamma \in \mathcal{L}} \text{length}(\gamma) = \inf_{\gamma \in \mathcal{L}} \int |\gamma'|$$

where $\mathcal{L}$ is the collection of piecewise smooth curves joining p, q.

Using Gauss' lemma (Exercise 24), one can show

Proposition 26. *Fix* $p \in M, r < \text{inj}_p(M)$*, then for any* v *with* $|v| < r$*,*

$$d_g(p, \exp_p(v)) = |v|.$$

Thus the shortest curve joining p, q must be a geodesic.

Definition 27 (completeness of a manifold)**.** (M, g) *is (geodesically) complete if* $\exp_p(v)$ *is well-defined for all* $p \in M, v \in T_pM$*. Or equivalently, all the geodesics are well-defined on* $\mathbb{R}$*.*

Definition 28 (completeness of a vector field)**.** X *is complete if it has a global flow, i.e. the integral curve extends to* $\mathbb{R}$*.*

Exercise 29. *(1) let* (M, g) *be complete,* V *a smooth vector field with* $|V| \leqslant C$*, show that* V *is complete.*

(2) let (M, g) *be complete, show that every Killing vector field is complete.*

Hopf-Rinow theorem

Theorem 30 (Hopf-Rinow)**.** *The followings are euqivalent*

(1) (M, g) *is geodesically complete;*

(2) there exists some $p \in M$ *such that* $\exp_p$ *is well-defined on* T_pM*;*

(3) every closed and bounded subset of M *is compact.*

(4) (M, d_g) *is metrically complete.*

Exercise 31. *(1) every compact manifold is complete;*

(2) if $(M, g_1), (M, g_2)$ satisfies $g_1 \geqslant g_2$ and (M, g_2) is complete, then (M, g_1) is also complete.

(3) a riemannian manifold is said to be homogeneous if the isometry group acts transitively. show that the homogeneous manifolds are complete.

(4) let $O \subset (M, g)$ be an open subset, show that if (O, g) is complete, then $O = M$.

(5) let $(M, g) = (\mathbb{R} \times N, \mathrm{d}r^2 + \rho^2 g_N)$ where $\rho : \mathbb{R} \to (0, \infty)$, (N, g_N) is complete. show that (M, g) is complete.

(6) show that any riemannian manifold (M, g) admts a conformal change $(M, \lambda^2 g)$ that is complete.

1.7. Normal coordinates

Definition 32 (normal coordinates)**.** *Take an ONB of T_pM, and define $B : \mathbb{R}^n \to T_pM, r \mapsto r^i e_i$, which is an isometry. The (reversed) map*

$$\varphi = B^{-1} \circ \exp_p^{-1} : U \to T_pM \to \mathbb{R}^n$$

gives $(x^i) = (r^i \circ \varphi)$, the normal coordinates centered at p.

List of properties:

- $\varphi_* \frac{\partial}{\partial x^i}|_p = \frac{\partial}{\partial r^i}$ and $\varphi_*(e_i) = B^{-1} e_i = \frac{\partial}{\partial r^i}$, so $\frac{\partial}{\partial x^i}|_p = e_i$;
- $g_{ij}(p) = \delta_{ij}$;
- for $v = v^i \frac{\partial}{\partial x^i}|_p$, $\gamma_v^i(t) = tv^i$;
- $\Gamma_{ij}^k|_p = 0$, thus $\frac{\partial}{\partial x^k} g_{ij}|_p = 0$.

Theorem 33 (local expansion of metric)**.** *Under any normal coordinates,*

$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{iklj}|_p x^k x^l + O(|x|^3), \quad g^{ij} = \delta_{ij} + \frac{1}{3} R_{iklj}|_p x^k x^l + O(|x|^3),$$

and also,

$$\det g = 1 - \frac{1}{3} \operatorname{Ric}_{ij}|_p x^i x^j + O(|x|^3), \quad \frac{\partial g_{ij}}{\partial x^k x^l} = \frac{1}{3}(R_{iklj}|_p + R_{ilkj}|_p).$$

Exercise 34. *show for small r that*

(1) $\operatorname{Vol}(B(p, r)) = \omega_n r^n \left(1 - \frac{S_p}{6(n+2)} r^2 + O(r^3)\right)$*;*

(2) $\mathrm{Area}(S(p,r)) = n\omega_n r^{n-1}\left(1 - \frac{S_p}{6n}r^2 + O(r^3)\right)$.

Consider the distance function $r(q) := d_g(p,q)$ on $U = M \backslash \mathrm{cut}(p)$.
List of properties:

- r is continuous and is smooth on $U\backslash\{p\}$;
- $r(q) = |\exp_p^{-1}(q)|$;
- $\nabla r = g^{ij}\frac{\partial r}{\partial x^i}\frac{\partial}{\partial x^j}$ is a smooth vector field on $U\backslash\{p\}$.

In normal coordinates, recall that $\gamma_v^i(t) = x^i \circ \gamma_v(t) = tv^i$ for $v = v^i\frac{\partial}{\partial x^i}|_p$, so $r(q) = |\exp_p^{-1}(q)| = |\exp_p^{-1}(\exp_p(x^i(q)\frac{\partial}{\partial x^i}|_p))| = \sqrt{\sum(x^i(q))^2}$.

Definition 35 (radial vector field)**.** $\partial_r := \frac{x^i}{r}\frac{\partial}{\partial x^i} = \sum_i \frac{\partial r}{\partial x^i}\frac{\partial}{\partial x^i}$.

Theorem 36. *On $U\backslash\{p\}$*

(1) ∂_r is nowhere-vanishing and orthogonal to the level set of r;

(2) (Gauss' lemma) $\nabla r = \partial_r, |\partial_r| = 1$.

List of properties: (as corollaries)

- $\mathcal{H}_r(\partial r) = \nabla_{\partial_r}\partial_r = 0$.
- $\sum_j g_{ij}x^j = x^i$, $g_{ij} = \delta_{ij} - \sum_k \frac{\partial g_{ik}}{\partial x^j}x^k$;
- $\sum_j \frac{\partial g_{ij}}{\partial x^k}x^j = \sum_j \frac{\partial g_{kj}}{\partial x^i}x^j$, $\sum_{i,j}\frac{\partial g_{ij}}{\partial x^k}x^i x^j = \sum_{i,j}\frac{\partial g_{jk}}{\partial x^i}x^i x^j = 0$
- $\sum_{i,j}\Gamma_{ij}^k x^i x^j = 0$.

Exercise 37. *consider the normal coordinates around p, show that at p*

$$\frac{\partial^2}{\partial x^l \partial x^k}g_{ji} + \frac{\partial^2}{\partial x^j \partial x^l}g_{ki} + \frac{\partial^2}{\partial x^k \partial x^j}g_{li} = 0.$$

Exercise 38. *show that in a riemannian manifold,*

$$d(\exp_p(v), \exp_p(w)) = |v - w| + O(r^2)$$

for $v, w \in T_pM, |v|, |w| \leqslant r$.

1.8. Hodge star operator and Hodge decomposition

Inner product

Definition 39 (musical operators)**.**

(1) $X^\flat := g_{ij}X^i\mathrm{d}x^j$; *(2)* $\omega^\sharp := g^{ij}\omega_i\frac{\partial}{\partial x^j}$

A natural way to extend g is $g(\mathrm{d}x^i, \mathrm{d}x^j)(= g((\mathrm{d}x^i)^\sharp, (\mathrm{d}x^j)^\sharp)) = g^{ij}$, or

$$g(\mathrm{d}x^I, \mathrm{d}x^J) = k!\det\begin{pmatrix} g^{i_1j_1} & \cdots & g^{i_1j_k} \\ \vdots & \ddots & \vdots \\ g^{i_kj_1} & \cdots & g^{i_kj_k} \end{pmatrix} =: k!g^{IJ}$$

for $\wedge^kT^*M$. For $\varphi = \sum f_{i_1\cdots i_k}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}$, we write

$$\varphi_{i_1\cdots i_k} = \sum_{\sigma\in S_k}(-1)^{|\sigma|}f_{i_{\sigma(1)}\cdots i_{\sigma(k)}}$$

where $\varphi_{i_1\cdots i_k}$ is skew-symmetric.

Definition 40 (inner product for k-forms)**.** *(1)* $\langle\varphi,\psi\rangle := \frac{1}{k!}g(\varphi,\psi)$*;*
(2) $(\varphi,\psi) := \int\langle\varphi,\psi\rangle\,\mathrm{d}\,\mathrm{Vol} = \frac{1}{k!}\int g(\varphi,\psi)\,\mathrm{d}\,\mathrm{Vol}$.

List of properties:

- $\varphi = \frac{1}{k!}\sum\varphi_{i_1\cdots i_k}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k} = \sum_{i_1<\cdots<i_k}\varphi_{i_1\cdots i_k}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}$;
- $\langle\varphi,\psi\rangle = g^{IJ}\varphi_I\psi_J = \frac{1}{k!}\sum g^{i_1j_1}\cdots g^{i_kj_k}\varphi_{i_1\cdots i_k}\psi_{j_1\cdots j_k}$;
- $\langle\mathrm{d}\,\mathrm{Vol},\mathrm{d}\,\mathrm{Vol}\rangle = 1$.

Exercise 41. *prove the properties above.*

Hodge star operator

Definition 42 (Hodge star operator)**.** *Take an ONB of* T^*M, $\xi^1\wedge\cdots\wedge\xi^n = \mathrm{d}\,\mathrm{Vol}_g$. *Define the linear operator* $* : \Omega^k(M)\to\Omega^{n-k}(M)$ *by*

$$*(v_I\xi^I) = v_I\,\mathrm{sgn}(I,I^c)\xi^{I^c}$$

where $I = (i_1\cdots i_k), I^c = (j_1\cdots j_{n-k}), i_1<\cdots<i_k, j_1<\cdots<j_{n-k}$.

List of properties:

- $*1 = \mathrm{d}\,\mathrm{Vol}_g, *\,\mathrm{d}\,\mathrm{Vol}_g = 1$, and $**\,v = (-1)^{k(n-k)}v$, for $v\in\Omega^k(M)$;
- $*(u\wedge v) = \langle *u, v\rangle = (-1)^{k(n-k)}\langle u, *v\rangle$, for $u\in\Omega^k(M), v\in\Omega^{n-k}(M)$;
- $u\wedge *v = v\wedge *u = \langle u,v\rangle\,\mathrm{d}\,\mathrm{Vol}_g, \langle *u, *v\rangle = \langle u,v\rangle$, for $u,v\in\Omega^k(M)$. Thus $(u,v) = \int u\wedge *v$.

Definition 43 (adjoint operator of d)**.** $(\mathrm{d}\varphi,\psi) =: (\varphi,\mathrm{d}^*\psi)$.

Theorem 44 (expression of d^*)**.** *On* $\Omega^k(M)$, $\mathrm{d}^* = (-1)^{nk+n+1} * \mathrm{d}*$.

Proof. For $u \in \Omega^{k-1}(M), v \in \Omega^k(M)$,

$$\begin{aligned}\int \langle u, *\,\mathrm{d} * v\rangle \,\mathrm{d}\,\mathrm{Vol}_g &= \int u \wedge **\,\mathrm{d} * v\\ &= (-1)^{(k-1)(n-k+1)} \int u \wedge \mathrm{d} * v\\ &\overset{*}{=} (-1)\cdot(-1)^{k-1}\cdot(-1)^{(k-1)(n-k+1)} \int \mathrm{d}u \wedge *v\\ &= (-1)^{nk+n+1} \int \langle \mathrm{d}u\,, v\rangle \,\mathrm{d}\,\mathrm{Vol}_g\,.\end{aligned}$$

Here we use Stokes' formula for $\overset{*}{=}$. □

Exercise 45. *for $\omega \in \Omega^p(M)$, show that*

$$(\mathrm{d}\omega)(X_0, \cdots, X_p) = \sum (-1)^i (\nabla_{X_i}\omega)(X_0, \cdots, \widehat{X_i}, \cdots, X_p).$$

Exercise 46. *for 1-form ω, show that*

$$\mathrm{d}^*\omega = -g^{ij}\left(\frac{\partial \omega_i}{\partial x^j} - \Gamma^k_{ij}\omega_k\right) =: -\nabla^i\omega_i.$$

Divergence

Definition 47 (divergence)**.** *The divergence of X is defined by*

$$\operatorname{div} X \cdot \mathrm{d}\,\mathrm{Vol}_g = L_X \mathrm{d}\,\mathrm{Vol}_g\,.$$

List of properties:

- $\operatorname{div} X = \frac{\partial X^i}{\partial x^i} + \Gamma^s_{is}X^i = \nabla_i X^i$ (regrad $\nabla_i X^j$ as coefficient of $\nabla_i X$);
- divergence theorem: if X is of compact support, then

$$\int \operatorname{div} X \,\mathrm{d}\,\mathrm{Vol}_g = 0.$$

- for 1-form ω with compact support, $\mathrm{d}^*\omega = \operatorname{div}\omega^\sharp$, so

$$\int \mathrm{d}^*\omega \,\mathrm{d}\,\mathrm{Vol}_g = 0.$$

- for $f_0, f_1 \in C_0^\infty(M)$, $\operatorname{div} f_1\nabla f_2 = g(\nabla f_1, \nabla f_2) + f_1\Delta f_2$, so

$$\int f_1 \Delta f_2 = -\int g(\nabla f_1, \nabla f_2) = \int f_2 \Delta f_1.$$

Exercise 48. *(1) solve Exercise 46 with the divergence theorem;*

(2) regard ∇X as $\nabla X^\flat$, then $\operatorname{div} X = \operatorname{tr}_g(\nabla X)$, this is a more general definition of divergence. for any smooth k-tensor field, define

$$\operatorname{div} F = \operatorname{tr}_g(\nabla F),$$

where the trace is taken on the first two indices. For smooth covariant k-tensor field F and $(k+1)$-tensor field on a compact manifold (M, g) with boundary, show that

$$\int_M \langle \nabla F, G \rangle \, \mathrm{d}\operatorname{Vol}_g = \int_{\partial M} \left\langle F \otimes N^\flat, G \right\rangle \mathrm{d}\operatorname{Vol}_{\widehat{g}} - \int_M \langle F, \operatorname{div} G \rangle \, \mathrm{d}\operatorname{Vol}_g$$

where $\widehat{g}$ is the induce metric of ∂M.

(3) let (M, g) be a riemannian manifold and $f : M \to \mathbb{R}$ a lipschitz function. then for any $\varphi \in C_0^\infty(M, \mathbb{R})$,

$$-\int_M \langle \nabla \varphi, \nabla f \rangle \, \mathrm{d}\operatorname{Vol}_g = \int_M \Delta_g \varphi \cdot f \, \mathrm{d}\operatorname{Vol}_g .$$

Hodge decomposition

Definition 49 (Beltrami-Laplace operator (a.k.a. Hodge laplacian))**.**

$$\Delta := \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$$

A k-form u is called harmonic if $\Delta u = 0$, denote by $\mathcal{H}^k(M)$ the set of harmonic k-forms.

Theorem 50 (Hodge decomposition)**.** *There is an orthogonal decomposition*

$$\Omega^k(M) = \mathcal{H}^k(M) \oplus \mathrm{d}(\Omega^{k-1}(M)) \oplus \mathrm{d}^*(\Omega^{k+1}(M)).$$

Moreover, $\dim_{\mathbb{R}} \mathcal{H}^k(M) < \infty$.

Theorem 51. *$\mathcal{H}^k(M) \cong H^k_{dR}(M; \mathbb{R})$.*

Exercise 52. *(1) show that $\Delta u = 0$ iff $\mathrm{d}u = 0, \mathrm{d}^* u = 0$;*

(2) prove Theorem 51;

(3) show that $H^1_{dR}(\mathbb{R}^2 \backslash \{0\}; \mathbb{R}) \neq 0$.

(4) suppose that M is connected, show that $H_{dR}(M, \mathbb{R}) \cong \mathbb{R}$.

Covraiant derivatives

A seemingly natural way to extend ∇ is using musical operators, i.e.

$$\nabla_{\frac{\partial}{\partial x^i}} \mathrm{d}x^j = \left(\nabla_{\frac{\partial}{\partial x^i}} (\mathrm{d}x^j)^\sharp\right)^\flat = \left(\nabla_{\frac{\partial}{\partial x^i}} g^{jk} \frac{\partial}{\partial x^k}\right)^\flat = -\Gamma^j_{ik} \mathrm{d}x^k.$$

But Leibniz rule simplifies the calculations greatly:

$$\left(\nabla_{\frac{\partial}{\partial x^i}} \mathrm{d}x^j\right) \frac{\partial}{\partial x^k} = \frac{\partial}{\partial x^i} \left\langle \mathrm{d}x^j, \frac{\partial}{\partial x^k} \right\rangle - \left\langle \mathrm{d}x^j, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} \right\rangle = -\Gamma^s_{ik}\delta_{js} = -\Gamma^j_{ik}.$$

Definition 53 (covraiant derivative)**.** *For $T \in \Gamma(M, \otimes^r T^*M \otimes \otimes^s TM)$, the covariant derivative $\nabla T \in \Gamma(M, \otimes^{r+1} T^*M \otimes \otimes^s TM)$ is defined by*

$$(\nabla T)(X, X_1, \cdots, \omega_s) = (\nabla_X T)(X_1, \cdots, \omega_s).$$

For $T = T^{j_1 \cdots j_s}_{i_1 \cdots i_r} \mathrm{d}x^{i_1} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_s}}$, $\nabla T = W^{j_1 \cdots j_s}_{i i_1 \cdots i_r} \mathrm{d}x^i \otimes \mathrm{d}x^{i_1} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_s}} =$

$$\left(\frac{\partial}{\partial x^i} T^{j_1 \cdots j_s}_{i_1 \cdots i_r} - \sum_{l=1}^{r} \Gamma^p_{i i_l} T^{j_1 \cdots j_s}_{i_1 \cdots p \cdots i_r} + \sum_{m=1}^{s} \Gamma^{j_m}_{iq} T^{j_1 \cdots q \cdots j_s}_{i_1 \cdots i_r}\right) \mathrm{d}x^i \otimes \mathrm{d}x^{i_1} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_s}}.$$

We ususlly write $T^{j_1 \cdots j_s}_{i_1 \cdots i_r}$, i.e. the coefficient, instead of the whole tensor.

Definition 54 (2nd covariant derivative)**.** $\nabla^2 T := \nabla(\nabla T)$*, or locally*

$$\nabla_k \nabla_i T^{j_1 \cdots j_s}_{i_1 \cdots i_r} = \nabla_k (W^{j_1 \cdots j_s}_{i i_1 \cdots i_r}).$$

Remark 55. *Caution! $(\nabla_k(\nabla_i T))^{j_1 \cdots j_s}_{i_1 \cdots i_r} \neq \nabla_k \nabla_i T^{j_1 \cdots j_s}_{i_1 \cdots i_r}$, in fact, the first one is not a tensor.*

Lemma 56. $\nabla^2_{X,Y} T = \nabla_X \nabla_Y T - \nabla_{\nabla_X Y} T$*, or locally*

$$\nabla_k \nabla_i T^{j_1 \cdots j_s}_{i_1 \cdots i_r} = (\nabla_k(\nabla_i T))^{j_1 \cdots j_s}_{i_1 \cdots i_r} - (\Gamma^j_{ki} \nabla_j T)^{j_1 \cdots j_s}_{i_1 \cdots i_r}.$$

Proof.

$$\begin{aligned}
\nabla_k (W^{j_1 \cdots j_s}_{i i_1 \cdots i_r}) &= \frac{\partial}{\partial x^k} W^{j_1 \cdots j_s}_{i i_1 \cdots i_r} + \sum_m \Gamma^{j_m}_{kq} W^{j_1 \cdots q \cdots j_s}_{i i_1 \cdots i_r} - \sum_l \Gamma^p_{k i_l} W^{j_1 \cdots j_s}_{i i_1 \cdots p \cdots i_r} \\
&\quad - \Gamma^j_{ki} W^{j_1 \cdots j_s}_{j i_1 \cdots p \cdots i_r} \\
&= \frac{\partial}{\partial x^k} (\nabla_i T)^{j_1 \cdots j_s}_{i_1 \cdots i_r} + \sum_m \Gamma^{j_m}_{kq} (\nabla_i T)^{j_1 \cdots q \cdots j_s}_{i_1 \cdots i_r} \\
&\quad - \sum_l \Gamma^p_{k i_l} (\nabla_i T)^{j_1 \cdots j_s}_{i_1 \cdots p \cdots i_r} - \Gamma^j_{ki} W^{j_1 \cdots j_s}_{j i_1 \cdots p \cdots i_r} \\
&= (\nabla_k(\nabla_i T))^{j_1 \cdots j_s}_{i_1 \cdots i_r} - (\Gamma^j_{ki} \nabla_j T)^{j_1 \cdots j_s}_{i_1 \cdots i_r}.
\end{aligned}$$

□

Ricci identity

From the definition of curvature tensor,

$$\begin{aligned}R(X,Y)T &= \nabla_X\nabla_Y T - \nabla_{\nabla_X Y}T - \nabla_Y\nabla_X T + \nabla_{\nabla_Y X}T\\ &= \nabla^2_{X,Y}T - \nabla^2_{Y,X}T.\end{aligned}$$

$$\begin{aligned}\nabla_k\nabla_l T^{j_1\cdots j_s}_{i_1\cdots i_r} - \nabla_l\nabla_k T^{j_1\cdots j_s}_{i_1\cdots i_r} &= \left(R\left(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\right)T\right)\left(\frac{\partial}{\partial x^{i_1}},\cdots,\mathrm{d}x^{j_s}\right)\\ &= \left(R\left(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\right)T\right)T^{j_1\cdots j_s}_{i_1\cdots i_r}\\ &\quad+\sum_m R^{j_m}_{klq}T^{j_1\cdots q\cdots j_s}_{i_1\cdots i_r} - \sum_t R^p_{kli_t}T^{j_1\cdots j_s}_{i_1\cdots p\cdots i_r}\end{aligned}$$

Since $R\left(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\right)f = 0$ for smooth function f, we obtain the following:

Theorem 57 (Ricci identity)**.**

$$\nabla_k\nabla_l T^{j_1\cdots j_s}_{i_1\cdots i_r} - \nabla_l\nabla_k T^{j_1\cdots j_s}_{i_1\cdots i_r} = \sum_m R^{j_m}_{klq}T^{j_1\cdots q\cdots j_s}_{i_1\cdots i_r} - \sum_t R^p_{kli_t}T^{j_1\cdots j_s}_{i_1\cdots p\cdots i_r}.$$

In particular, for vector fields and 1-forms,

$$\nabla_k\nabla_l X^i - \nabla_l\nabla_k X^i = R^i_{klq}X^q,$$

$$\nabla_k\nabla_l\omega_j - \nabla_l\nabla_k\omega_j = -R^p_{klj}\omega_p.$$

Exercise 58. *prove the ricci identity in (normal) local coordinates.*

Contraction and 2nd Bianchi identity

Using Leibniz rule for 2-tensor T,

$$Xg(g,T) = g(\nabla_X g, T) + g(g,\nabla_X T) = g(g,\nabla_X T),$$

this works similarly for 4-tensor S,

$$Xg(g\otimes g, S) = g(\nabla_X g\otimes g, S) + g(g\otimes g, \nabla_X T) = g(g\otimes g,\nabla_X T).$$

Proposition 59 (magic formulae for 2- and 4-tensors)**.**

$$\nabla_k g^{ij}T_{ij} = g^{ij}\nabla_k T_{ij},$$

$$\nabla_s g^{ij}g^{kl}S_{ijkl} = g^{ij}g^{kl}\nabla_s S_{ijkl}.$$

Theorem 60 (2nd Bianchi identity)**.**

$$\nabla_i R_{jkpq} + \nabla_j R_{kipq} + \nabla_k R_{ijpq} = 0.$$

As a corollary,

$$
\begin{aligned}
0 &= g^{jp}g^{kq}\left(\nabla_i R_{jkpq} + \nabla_j R_{kipq} + \nabla_k R_{ijpq}\right) \\
&= -\nabla_i g^{jp}g^{kq}R_{kjpq} + g^{jp}\nabla_j g^{kq}R_{ikqp} + g^{kq}\nabla_k g^{jp}R_{ijpq} \\
&= -\nabla_i S + g^{jp}\nabla_j \operatorname{Ric}_{ip} + g^{kq}\nabla_k \operatorname{Ric}_{iq},
\end{aligned}
$$

i.e. $\nabla_i S = 2g^{jk}\nabla_j \operatorname{Ric}_{ik}$, this is the contracted Bianchi identity.

Theorem 61 (Schur's lemma)**.** *Let (M, g) be a connected Riemannian manifold with $\dim M \geqslant 3$. If $f \in C^\infty(M)$, and one of the followings hold*

(1) $K = f$, i.e. $R(X,Y,Y,X) = |X \wedge Y|^2 f$ for $X, Y \in TM$;

(2) $\operatorname{Ric} = (n-1)fg$

then f is a constant.

Proof. Assuming *(2)*, $S = g^{ij}\operatorname{Ric}_{ij} = n(n-1)f$.

$$\nabla_k S = 2g^{ij}\nabla_i \operatorname{Ric}_{kj} = 2(n-1)g^{ij}\nabla_i f g_{kj} = 2(n-1)\nabla_k f.$$

Thus $n(n-1)\nabla_k f = 2(n-1)\nabla_k f$, which implies that f is constant. □

Exercise 62. *prove the 2nd Bianchi identity in local coordinates.*

1.10. Miscellany

Riemannian submersions

Exercise 63. *let $\pi : (\overline{M}, \overline{g}) \to (M, g)$ be a riemannian submersion.*

(1) let $H \subset T\overline{M}$ be the subbundle such that $H_p \perp \ker \pi_{*,p}$,

- *(a) for each $X \in \Gamma(M, TM)$, there exists a unique $\overline{X} \in \Gamma(\overline{M}, H)$ such that $\pi_* \overline{X} = X$;*
- *(b) let $\sigma : [a,b] \to \overline{M}$ be a smooth curve, then for each $p \in \pi^{-1}(\sigma(a))$, there exists $\varepsilon > 0$ and a unique smooth curve $\overline{\sigma} : [a, a+\varepsilon] \to \overline{M}$ such that*

$$\overline{\sigma}(a) = p, \pi \circ \overline{\sigma} = \sigma, \overline{\sigma}'(t) \in H_{\overline{\sigma}(t)}.$$

(2) for $X, Y \in \Gamma(M, TM)$, we have

$$\nabla^{g}_{\overline{X}}\overline{Y} = \overline{\nabla^{h}_{X}Y} + \frac{1}{2}[\overline{X}, \overline{Y}]^v$$

where Z^v is the orthogonal projection of Z to $\ker \pi_*$.

(3) for $X, Y \in \Gamma(M, TM)$*, we have*

$$R(X,Y,Y,X) = \overline{R}(\overline{X},\overline{Y},\overline{Y},\overline{X}) + \frac{3}{4}\left|[\overline{X},\overline{Y}]^v\right|^2.$$

(4) show that $\pi \circ \exp_p(v) = \exp_{\pi(p)}(\mathrm{d}\pi_p(v))$*. in particular, if* $\widetilde{\gamma}$ *is a geodesic, then* $\pi \circ \widetilde{\gamma}$ *is a geodesic.*

(5) show that

(a) (M, g) *is complete if* $(\overline{M}, \overline{g})$ *is complete;*

(b) π *is a fibration if* $(\overline{M}, \overline{g})$ *is complete.*

(c) give a counterexample when $(\overline{M}, \overline{g})$ *is not complete.*

Lie groups

A Riemannian metric h on a Lie group G is said to be left-invariant if $L_g^* h = h$, and bi-invariant if both left- and right-invariant.

Exercise 64. *let* G *be a lie group with* $\mathfrak{g}$ *the lie algebra.*

(1) if h *is a bi-invariant metric on a Lie group* G*, show that for left-invariant vector fields* X, Y, Z

$$h([X,Y],Z) = h(X,[Y,Z]).$$

(2) let $\langle \bullet, \bullet \rangle_e$ *be an inner product on* $\mathfrak{g}$*, define*

$$\langle X_g, Y_g \rangle = \langle (L_{g^{-1}})_* X_g, (L_{g^{-1}})_* Y_g \rangle_e.$$

show that

(a) $\langle \bullet, \bullet \rangle$ *is a left-invariant Riemannian metric on* G*.*

(b) there is a bijection

$$\{\textit{Inner products on } \mathfrak{g}\} \longleftrightarrow \left\{\begin{matrix}\textit{left-invariant} \\ \textit{metrics on } G\end{matrix}\right\}.$$

(c) under the above bijection, $\mathrm{Ad}(G)$*-invariant inner products on* $\mathfrak{g}$ *correspond to bi-invariant riemannian metrics on* G*.*

(3) let h *be a bi-invariant riemannian metric with connection* ∇*, then*

$$\nabla_X Y = \frac{1}{2}[X,Y],$$

for left-invariant vector fields X, Y*. Moreover,*

$$R(X,Y,Z,W) = -\frac{1}{4}([X,Y],[Z,W]),$$

for left-invariant vector fields X, Y, Z, W*.*

(4) *let h be a bi-invariant riemannian metric. show that*

   (a) *the geodesics on G are precisely the integral curves of the left-invariant vector fields.*

   (b) *the exponential map for the lie group coincides with the exponential map of the levi-civita connection.*

Exercise 65. *the heisenberg group with its lie algebra is*

$$G = \left\{ \begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}, \quad \mathfrak{g} = \left\{ \begin{pmatrix} & x & z \\ & & y \\ & & \end{pmatrix} \middle| \, x, y, z \in \mathbb{R} \right\}.$$

a basis for the lie algebra is

$$X = \begin{pmatrix} & 1 & \\ & & \\ & & \end{pmatrix}, Y = \begin{pmatrix} & & \\ & & 1 \\ & & \end{pmatrix}, Z = \begin{pmatrix} & & 1 \\ & & \\ & & \end{pmatrix}.$$

(1) *show that the only non-zero brackets are $[X, Y] = -[Y, X] = Z$.*

(2) *consider a left-invariant metric with $\{X, Y, Z\}$ an onb. show that the ricci tensor has both negative and positive eigenvalues.*

(3) *show that the scalar curvature is constant.*

(4) *show that the ricci tensor is not parallel.*

2. The Bochner technique

2.1. Killing vector fields

Bochner formula for smooth functions

Proposition 66. *Let $f : M \to \mathbb{R}$ be a smooth function over (M, g), then*

$$\frac{1}{2}\Delta_g|\nabla f|^2 = |\operatorname{Hess} f|^2 + \operatorname{Ric}(\nabla f, \nabla f) + g(\nabla\Delta_g f, \nabla f).$$

Curvature and Killing vector fields

Definition 67 (Killing vector field)**.** *$L_X g = 0$ (the flow is isometric).*

Using Koszul formula, we can show

$$g((L_X\nabla)_Y Z, W) = 0, \text{ i.e. } L_X\nabla = 0.$$

which gives a useful relation

$$R(X,Y)Z + \nabla^2_{Y,Z}X = 0.$$

It can also be stated and proven in terms of coefficients.

$$g_{il}\nabla_j\nabla_k X^i + R_{ijkl}X^i = 0.$$

Theorem 68. *Let X be a Killing vector field, $f = \frac{1}{2}|X|^2$,*

(1) $\nabla f = -\nabla_X X$;

(2) For any vector field V,

$$\operatorname{Hess} f(V,V) = g(\nabla_V X, \nabla_V X) - R(V,X,X,V).$$

In particular,

$$\Delta_g f = |\nabla X|^2 - \operatorname{Ric}(X,X).$$

Theorem 69. *Let (M,g) be a compact Riemannian manifold*

(1) if $\operatorname{Ric} < 0$*, then M has no non-trivial Killing vector field.*

(2) (Bochner) if $\operatorname{Ric} \leqslant 0$, then a vector field is parallel iff it is Killing.

The following theorem is proven using "linear algebra".

Theorem 70. *Let (M,g) be a compact Riemannian manifold with positive sectional curvature. If M is of even dimension, then every Killing field has a zero.*

Remark 71. *There are examples of non-vanishing Killing vector fields if M is odd, e.g. $V_x = (x_2, -x_1, \cdots, x_{2n}, -x_{2n-1})$ on S^{2n-1}.*

Exercise 72 (conformal killing vector field)**.** *a vector field X is a conformal killing vector field if $L_X g = fg$ for some smooth function $f : M \to \mathbb{R}$.*

(1) show that $f = 2\operatorname{div} X$.

(2) show that

$$\frac{1}{2}\Delta_g|X|^2 = |\nabla X|^2 - \operatorname{Ric}(X,X) - \left(1 - \frac{2}{n}\right)\langle \nabla \operatorname{div} X, X\rangle.$$

(3) let (M,g) be a closed Riemannian manifold with $\operatorname{Ric} < 0$*, show that there are no non-zero conformal killing fields.*

2.2. Harmonic 1-forms

Bochner formula for harmonic 1-forms

Proposition 73. *Let (M, g) be a compact Riemannian manifold, $\alpha \in \Omega^1(M)$ be a harmonic form, then*

$$\frac{1}{2}\Delta_g|\alpha|^2 = |\nabla\alpha|^2 + \mathrm{Ric}(\alpha^\sharp, \alpha^\sharp).$$

For general 1-form α, the Bochner formula is

$$\frac{1}{2}\Delta_g|\alpha|^2 = -g(\Delta\alpha, \alpha) + |\nabla\alpha|^2 + \mathrm{Ric}(\alpha^\sharp, \alpha^\sharp).$$

where Δ is the Hodge laplacian.

Theorem 74. *Suppose (M, g) is a compact Riemannian manifold of non-negative Ricci curvature.*

(1) Every harmonic 1-form is parallel. Hence $b_1(M) \leqslant \dim M$.

(2) If $\mathrm{Ric} > 0$, then $b_1(M) = 0$.

2.3. Smooth maps

Proposition 75. *Let $f : (M, g) \to (N, h)$ be a smooth map, then*

$$\begin{aligned}\frac{1}{2}\nabla_g|\mathrm{d}f|^2 = (\widehat{\nabla}\Delta f, \mathrm{d}f) + |\widetilde{\nabla}\mathrm{d}f|^2 + g^{ik}g^{jl}h_{\alpha\beta}\,\mathrm{Ric}_{ij}\, f_k^\alpha f_l^\beta \\ - g^{ij}g^{kl}R_{\alpha\beta\gamma\delta}f_i^\alpha f_j^\delta f_k^\beta f_l^\gamma.\end{aligned}$$

3. Jacobi fields

3.1. Variation formulae and Jacobi fields

Variations

Fix $p, q \in (M, g), a < b \in \mathbb{R}$, let $\mathcal{L}$ be the space of smooth curves $\gamma : [a, b] \to M$ with $\gamma(a) = p, \gamma(b) = q$.

Definition 76 (energy)**.** *For $\gamma \in \mathcal{L}$, $E(\gamma) := \int_a^b \left|\gamma_* \frac{\mathrm{d}}{\mathrm{d}t}\right|^2 \mathrm{d}t$.*

Definition 77 (proper variation)**.** *A proper variation of γ is a smooth map $\alpha : [a, b] \times (-\varepsilon, \varepsilon) \to M$ with $\alpha(\cdot, s) \in \mathcal{L}, \alpha(\cdot, 0) = \gamma$.*

Proposition 78 (definition of variational field). *Let $X \in \Gamma([a,b], \gamma^*TM)$ with $X_a = X_b = 0$, then there exists a proper variation α of γ with*

$$\alpha_* \frac{\partial}{\partial s}\bigg|_{s=0} = X.$$

X is called the variational vector field of α.

Theorem 79 (1st variation formula). *Let α be a proper variation of γ with V the variational vector field, then*

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} E(\alpha(\cdot, s)) = \int_a^b \left\langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma' \right\rangle \mathrm{d}t = -\int_a^b \left\langle V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma' \right\rangle \mathrm{d}t\,.$$

We can similarly consider the 2nd variation: $\alpha(t, s_1, s_2) : [a,b] \times (-\varepsilon_1, \varepsilon_1) \times (-\varepsilon_2, \varepsilon_2) \to M, \alpha(t,0,0) = \gamma(t)$ with variational fields

$$\alpha_* \frac{\partial}{\partial s_1}\bigg|_{s_1=s_2=0} = V,\ \alpha_* \frac{\partial}{\partial s_2}\bigg|_{s_1=s_2=0} = W.$$

Theorem 80 (2nd variation formula). *Let α be a proper 2nd variation with V, W the variational vector fields.*

$$\begin{aligned}\frac{\partial^2}{\partial s_1 \partial s_2}\bigg|_{s_1=s_2=0} E(\alpha(\cdot, s_1, s_2)) &= \int_a^b \left\langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W \right\rangle \mathrm{d}t \\ &\quad - \int_a^b R(V, \gamma', \gamma', W)\,\mathrm{d}t \\ &\quad - \int_a^b \left\langle \left(\overline{\nabla}_{\frac{\partial}{\partial s_1}} \alpha_* \frac{\partial}{\partial s_2}\right)\bigg|_{s_1=s_2=0}, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma' \right\rangle \mathrm{d}t\,.\end{aligned}$$

Remark 81. *An important case is when s_1, s_2 coincide, which occurs in the proof of Synge and Weinstein-Synge theorems.*

Jacobi fields

Definition 82 (Jacobi field). *Let $\gamma : [a,b] \to (M,g)$ be a geodesic. A vector field J along γ is called a Jacobi field if*

$$\widehat{\nabla}\widehat{\nabla} J + R(J, \gamma')\gamma' = 0.$$

Proposition 83 (local expansion of the length). *Let $f(t) = |J|^2$, where J is a Jacobi field along a geodesic γ, then*

$$f(t) = t^2 - \frac{1}{3} R(J', \gamma', \gamma', J')|_0 t^4 + O(t^6).$$

Acturally, Proposition 83 implies Theorem 33.

Theorem 84 (characterization of a Jacobi field)**.** *Every Jacobi field is given by some variation along some geodesic. Let* (M, g) *be a Riemannian manifold,* $\gamma : [0, 1] \to M$ *be a geodesic, then the Jacobi field along* γ *with* $J(0) = 0$ *and* $J'(0) = v$ *is given by*

$$J = \alpha_* \frac{\partial}{\partial s}\bigg|_{s=0}, \quad \alpha = \exp_{\gamma(0)}(t(\gamma'(0) + sv))$$

for s *small enough. In particular,*

$$J(t) = (\exp_{\gamma(0)})_{*,t\gamma'(0)}(tv).$$

The following result can be proved using normal coordinates.

Proposition 85. *Let* (M, g) *be a complete Riemannian manifold,* $p \in M, \gamma : [0, b] \to M \setminus \mathrm{cut}(p)$ *a unit-speed geodesic with* $\gamma(0) = p$*, and* r *the distance from* p*. If* J *is a normal Jacobi field along* γ *with* $J(0) = 0$*, then*

$$\mathcal{H}_r(J(t)) = J'(t), \quad \mathcal{H}(\gamma'(t)) = 0.$$

In particular,

$$\mathrm{Hess}\, r(J, W)|_s = \int_0^s \langle J', W' \rangle - R(J, \gamma', \gamma', W)\, \mathrm{d}t\,,$$

for any vector field W *along* γ *with* $W(0) = 0$*.*

Exercise 86. *let* $\sigma : (-\varepsilon, \varepsilon) \to (M, g)$ *be a smooth curve and* $V(s) \in \Gamma((-\varepsilon, \varepsilon), \sigma^*TM)$*. consider*

$$\alpha(t, s) = \exp_{\sigma(s)}(tV(s)).$$

compute the variational vector field $W(t) = \alpha_* \frac{\partial}{\partial s}\big|_{s=0}$ *and point out* $W(0), \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W(0)$*.*

3.2. Conjugate loci and cut loci

Definition 87 (conjugate locus)**.** *Let* $\gamma : I \to (M, g)$ *be a geodesic with* $p = \gamma(a), q = \gamma(b)$*. We say* p, q *are conjugate along* γ *if there is a non-trivial Jacobi field along* γ *with* $J(a) = J(b) = 0$*. Write the cut locus* $\mathrm{conj}(p)$ *for the set of all conjugate points of* p *along some geodesic.*

Theorem 88. *Let* $v \in \mathcal{E}_p, \gamma_v(t) = \exp_p(tv), q = \gamma_v(1)$*, then* v *is a critical point of* $\exp_p : \mathcal{E}_p \to M$ *iff* q *is conjugate to* p *along* γ_v*.*

Definition 89 (cut time, cut locus)**.** *Define the cut time of* (p, v) *by*

$$t_{\text{cut}}(p, v) = \sup\{b \,|\, \gamma_v|_{[0,b]} \textit{ is a minimal geodesic}\},$$

and the cut point along γ_v *by* $\gamma_v(t_{\text{cut}}(p, v))$*. Define the cut locus* $\text{cut}(p)$ *by the set of all cut points of* p*.*

Theorem 90. *Let* (M, g) *be a complete Riemannian manifold,* $p \in M, v \in T_pM$ *with* $|v| = 1$*, and* $c = t_{\text{cut}}(p, v)$*.*

(1) If $0 < b < c$*, then* $\gamma_v|_{[0,b]}$ *has no conjugate points and is the unique minimal unit-speed geodesic between* p *and* $\gamma_v(b)$*.*

(2) if $c < \infty$*, then* $\gamma_v|_{[0,c]}$ *is minimal. One or both of the followings hold:*

(a) $\gamma_v(c)$ *is conjugate to* p *along* γ_v*;*

(b) there are two or more unit-speed geodesics between p *and* $\gamma_v(c)$*.*

Example 91. *(1) For* $p \in S^n$, $\text{conj}(p) = \text{cut}(p) = \{-p\}$*.*

(2) For $p \in \mathbb{RP}^n$, $\text{conj}(p) = \{p\}$, $\text{cut}(p) \simeq S^{n-1}$*.*

(3) For $p = (x, y) \in S^1 \times \mathbb{R}$, $\text{conj}(p) = \emptyset$, $\text{cut}(p) = \{-x\} \times \mathbb{R}$*.*

(4) For $p \in \mathbb{T}^n$, $\text{cut}(p) \simeq \partial([0, 1]^n)$*.*

Exercise 92. *let* (M, g) *be a complete Riemannian manifold,* $p \in M$*. suppose there exists some* $q \in \text{cut}(p)$ *with* $d(p, q) = d(p, \text{cut}(p))$*.*

(1) show that either q *is conjugate to* p*, or there are exactly two unit-speed minimal geodesics* $\gamma_1, \gamma_2 : [0, b] \to M$ *between* p *and* q *with* $\gamma_1'(b) = -\gamma_2'(b)$*, where* $b = d(p, q)$*.*

(2) if $\text{inj}_p(M) = \text{inj}(M)$*, and* q *is not conjugate to* p *along any minimal geodesic, show that there is a closed unit-speed geodesic* $\gamma : [0, 2b] \to M$ *such* $\gamma(0) = \gamma(2b) = p$ *and* $\gamma(b) = q$*, where* $b = d(p, q)$*.*

There are many related topics like Morse index theorem, skeleton and cellular structure given by Morse theory, etc. To be added someday.

4. Curvature and topology

4.1. Spaces of non-positive sectional curvature

Theorem 93 (Cartan-Hadamard)**.** *Let* (M, g) *be a complete Riemannian manifold with non-positive sectional curvature. For any* $p \in M$, $\exp_p : T_pM \to M$ *is a covering map. The universal covering* $\widetilde{M} \cong \mathbb{R}^n$*.*

Corollary 94. *Suppose M, N are compact smooth manifolds. If one of them is simply-connected, then $M \times N$ does not admit a Riemannian metric with non-positive sectional curvature.*

Theorem 95 (characterization of CH manifolds)**.** *Let (M, g) be a simply-connected complete manifold. The followings are euqivalent.*

(1) M has non-positive sectional curvature;

(2) The differential of exponential map is length increasing, i.e.

$$|(\exp_p)_{*,v}(\widetilde{v})| \geqslant |\widetilde{v}|$$

for all $p \in M, v, \widetilde{v} \in T_pM$.

(3) The exponential map is distance increasing, i.e.

$$d_g(\exp_p(v), \exp_p(\widetilde{v})) \geqslant |v - \widetilde{v}|$$

for all $p \in M, v, \widetilde{v} \in T_pM$.

Moreover, if the conditions are satisfied, then the exponential map is diffeomorphic.

Exercise 96. *let (M, g) be a ch manifold, $p \in M$.*

(1) fix $v, \widetilde{v} \in T_pM$, show that for $0 < t \leqslant T$,

$$|v - \widetilde{v}| \leqslant \frac{d(\exp_p(tv), \exp_p(t\widetilde{v}))}{t} \leqslant \frac{d(\exp_p(Tv), \exp_p(T\widetilde{v}))}{T}.$$

(2) let $f(x) = \frac{1}{2}d(x, p)^2$, show that f is strictly geodesically convex, i.e. for any non-trivial geodesic $\gamma : [0, 1] \to M$,

$$f(\gamma(t)) < (1 - t)f(\gamma(0)) + tf(\gamma(1)).$$

Theorem 97 (Cartan)**.** *Let (M, g) be a CH manifold, G a compact Lie group acting smoothly and isometrically on M, then G has a fixed point.*

Theorem 98 (Cartan)**.** *Let (M, g) be a complete Riemannian manifold with non-positive sectional curvature, then $\pi_1(M)$ is torsion free.*

4.2. Spaces of negative sectional curvature

Proposition 99. *Let (M, g) be a complete Riemannian manifold with non-positive sectional curvature and $\pi : \widetilde{M} \to M$ the universal covering. If $\widetilde{\gamma} : \mathbb{R} \to \widetilde{M}$ is a common axis for all elements of $\mathrm{Aut}_\pi(\widetilde{M})$, then M is not compact.*

Exercise 100. *let* (M, g) *be a closed riemannian manifold of dimension* $\geqslant 2$ *with negative sectional curvature. let* $\widetilde{M}$ *be its universal,* $\Gamma = \pi_1(M)$ *can be identified as a subgroup of* $\operatorname{Isom}(\widetilde{M})$ *by deck transformations.*

(1) show that there are $\gamma_1, \gamma_2 \in \pi_1(M)$ *with different axes.*

(2) show that the centralizer of $\Gamma \subset \operatorname{Isom}(\widetilde{M})$ *is trivial.*

Theorem 101 (Preissmann)**.** *Let* (M, g) *be a compact Riemannian manifold with negative sectional curvature.*

(1) Any non-trivial abelian subgroup of $\pi_1(M)$ *is isomorphic to* Z*.*

(2) $\pi_1(M)$ *is not abelian.*

Corollary 102. *Suppose* M, N *are compact cmooth manifolds. Then* $M \times N$ *does not admit a Riemannian metric of negative sectional curvature.*

Theorem 103. *Let* (M, g) *be a compact Riemannian manifold with negative sectional curvature.*

(1) (Byers) Any non-trivial solvable subgroups of $\pi_1(M)$ *is isometric to* $\mathbb{Z}$*. In particular,* $\pi_1(M)$ *is not solvable.*

(2) Any subgroup of $\pi_1(M)$ *which contains a non-trivial abelian normal subgroup is isomorphic to* $\mathbb{Z}$*.*

There are many further topics like Milnor's exponential-growth of fundamental group, CAT($\leqslant 0$) geometry, etc. To be added someday.

4.3. Spaces of non-negative curvature

Theorem 104 (Myers)**.** *Let* (M^n, g) *be a complete manifold. If*

$$\operatorname{Ric} \geqslant \frac{(n-1)g}{R^2}$$

then $\operatorname{diam}(M, g) \leqslant \pi R$*. In particular,* M *is compact and* $\pi_1(M)$ *is finite.* (Cheng) If $\operatorname{diam}(M, g) = \pi R$, then M is isometric to (S^n, g_{can}).

Exercise 105. *for* $(\mathbb{R}^2, g_a = e^{a(x^2+y^2)}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y))$*,*

(1) compute the curvatures, conclude that it is Einstein;

(2) show that if $a \geqslant 0$*, then it is complete;*

(3) show that if $a < 0$*, then it is not complete.*

Theorem 106 (Synge)**.** *Let (M, g) be a compact Riemannian manifold with positive sectional curvature.*

(1) If $\dim M$ is even and M is orientable, then M is simply connected;

(2) If $\dim M$ is odd, then M is orientable.

Corollary 107. *Let (M, g) be a compact Riemannian manifold with positive sectional curvature. If $\dim M$ is even and M is not orientable, then $\pi_1(M) = \mathbb{Z}/2\mathbb{Z}$.*

For example, $\mathbb{RP}^2 \times \mathbb{RP}^2, U(2), U(2)/O(2)$ do not admit a Riemannian metric with positive sectional curvature, in each case, the obstruction is the fundamental group.

Theorem 108 (Weinstein-Synge)**.** *Let (M^n, g) be a compact Riemannian manifold with positive sectional curvature. Given an isometry $F : M \to M$ such that F preserve the orientation if n is even, changes the orientation if n is odd. Then F has a fixed point.*

Exercise 109. *show that there is no compact manifold that admits both a metric of positive definite ricci curvature and a metric of non-positive sectional curvature.*

4.4. Space forms

Theorem 110 (Riemann-Hopf-Killing)**.** *Let (M, g) be a complete manifold with constant sectional curvature, then it is isometric to a Riemannian quotient of the form $\widetilde{M}/\Gamma$, where $\widetilde{M}$ is one of the models spaces*

(1) $\mathbb{R}^n$, *(2) $S^n(r)$,* *(3) $\mathbb{H}^n(r)$*

and $\Gamma \subset \mathrm{Isom}(\widetilde{M})$ is discrete and acts freely.

Here is a corollary of the Cartan-Ambrose-Hicks theorem.

Theorem 111. *Let (M, g_M) be connected, φ, ψ be two local isometries from M to (N, g_N). If there exists some point $p \in M$ with $\varphi(p) = \psi(p)$ and $\varphi_{*,p} = \psi_{*,p}$, then $\varphi = \psi$.*

Corollary 112. *Let (M, g) be a connected simply-connected complete Riemannian manifold. The followings are equivalent.*

(1) (M, g) is of constant sectional curvature.

(2) For every pair of points $p, q \in M$ and linear isometry $\Pi : T_pM \to T_pM$, there exists an isometry $\varphi : M \to M$ with $\varphi(p) = q, \varphi_{*,p} = \Phi$.

Corollary 113. *Let (M, g) be a complete and of constant sectional curvature 1. If $\dim M = 2m$, then (M, g) is isometric to S^{2m} or $\mathbb{RP}^{2m}$.*

For convenience, we write $\mathbb{S}_k^n$ for the n-dimensional space form with constant sectional curvature k, and

$$\mathrm{sn}_k(t) = \begin{cases} t & , \text{ if k=0} \\ \frac{1}{\sqrt{k}} \sin \sqrt{k} t & , \text{ if } k > 0 \\ \frac{1}{\sqrt{-k}} \sinh \sqrt{-k} t & , \text{ if } k < 0 \end{cases} .$$

Theorem 114 (Jacobi fields in space forms)**.** *Let (M, g) be a Riemannian manifold with constant sectional curvature k, and γ a unit-speed geodesic. Then a normal Jacobi field J with $J(0) = 0$ is of the form*

$$J(t) = a\,\mathrm{sn}_k(t)E(t),$$

where a is constant, $E(t)$ is any unit parallel vector field with $\langle E, \gamma' \rangle = 0$.

Theorem 115. *Let U be a geodesic ball around $p \in \mathbb{S}_k^n$, r the distance from p. Then on $U \backslash \{p\}$ under the normal coordinates,*

$$g = \mathrm{d}r^2 + \mathrm{sn}_k^2(r)\widehat{g},$$

where $\widehat{g}$ is the induced form on $U \backslash \{p\}$ by local trivialization.

Corollary 116 (an integral formula)**.** *Let U be a geodesic ball of radius b around $p \in \mathbb{S}_k^n$. If $f : U \to \mathbb{R}$ is a bounded integrable function, then*

$$\int_U f \,\mathrm{d}V_g = \int_{S^{n-1}} \int_0^b f \circ \Phi(\rho, \omega)\, \mathrm{sn}_k(\rho)^{n-1} \,\mathrm{d}\rho \,\mathrm{d}\,\mathrm{Vol}_{S^{n-1}} ,$$

where $\Phi : \mathbb{R}^+ \times S^{n-1} \to U \backslash \{p\}, (\rho, \omega) \mapsto \rho\omega$.

Remark 117. *A more general integral formula applies to the Heintze-Karcher type inequality for embedded hypersurfaces in space forms.*

Proposition 118. *Let U be a geodesic ball of radius b around $p \in \mathbb{S}_k^n$, r the distance from p. Then*

$$\mathcal{H}_r = \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)} \pi_r,$$

where π_r is the projection to the orthogonal complement of $\partial_r|_q$. Hence

$$\mathrm{Hess}\, r = \mathrm{sn}_k'(r)\, \mathrm{sn}_k(r)\widehat{g},$$

and

$$\Delta_g r = (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}, \quad \Delta_g r^2 = 2 + 2(n-1)r \cdot \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}.$$

5. Comparison theorems of curvatures

5.1. Rauch comparison

Rauch comparison and corollaries

Theorem 119 (Rauch comparison)**.** *Let $(M,g),(\widetilde{M},\widetilde{g})$ be two Riemannian manifolds with $\dim M \leqslant \dim \widetilde{M}$. Suppose that $\gamma,\widetilde{\gamma}:[0,l]\to M,\widetilde{M}$ are unit-speed geodesics, and*

(1) for any t and any planes $\Sigma,\widetilde{\Sigma} \subseteq T_{\gamma(t)}M, T_{\widetilde{\gamma}(t)}\widetilde{M}$ with $\gamma'(t),\widetilde{\gamma}'(t) \in \Sigma,\widetilde{\Sigma}$, the sectional curvatures satisfy

$$K_\Sigma(\gamma(t)) \leqslant \widetilde{K}_{\widetilde{\Sigma}}(\widetilde{\gamma}(t)),$$

(2) $\widetilde{\gamma}(0)$ has no conjugate points along $\widetilde{\gamma}|_{(0,l]}$.

Then for any Jacobi fields $J,\widetilde{J}$ along $\gamma,\widetilde{\gamma}$ with initial conditions $J(0) = c\gamma'(0), \widetilde{J}(0) = c\widetilde{\gamma}'(0), |J'(0)| = |\widetilde{J}'(0)|, g(J'(0),\gamma'(0)) = \widetilde{g}(\widetilde{J}'(0),\widetilde{\gamma}'(0))$, we have $|\widetilde{J}| \leqslant |J(t)|$ for all $t \in [0,l]$.

A useful case is when $(\widetilde{M},\widetilde{g})$ is the space form.

Corollary 120 (Jacobi field comparison)**.** *Let (M,g) be a complete Riemannian manifold, $p \in M, U = M\setminus \mathrm{cut}(p)$.Let $\gamma:[0,b]\to U$ be a unit-speed geodesic with $\gamma(0)=p$ and J be any normal Jacobi field along γ with $J(0)=0$. Then*

(1) if the sectional curvature $K_M \leqslant k$, then

$$|J(t)| \geqslant \mathrm{sn}_k(t)|J'(0)|$$

(2) if the sectional curvature $K_M \geqslant k$, then

$$|J(t)| \leqslant \mathrm{sn}_k(t)|J'(0)|$$

for all $t\in[0,b_1]$, where $b_1 = \begin{cases} b & , \text{ if } k \leqslant 0 \\ \min\{b,\pi R\} & , \text{ if } k = \frac{1}{R^2} > 0 \end{cases}$.

Corollary 121 (conjugate comparison)**.** *Let (M,g) be a complete Riemannian manifold with sectional curvature $K_M \leqslant k$.*

(1) If $k \leqslant 0$, then M has no conjugate points along any geodesic.

(2) If $k = \frac{1}{R^2} > 0$, then there is no conjugate point along any geodesic shorter that πR.

Corollary 122. *Let (M, g) be a complete Riemannian manifold. Suppose $0 < C_1 \leqslant K_M \leqslant C_2$, let γ be any geodesic in M and l be the distance along γ between two consecutive conjugate points on γ, then*

$$\frac{\pi}{\sqrt{C_2}} \leqslant l \leqslant \frac{\pi}{\sqrt{C_1}}.$$

In particular, $\exp_p$ *has no critical points on* $B\left(0, \frac{\pi}{\sqrt{C_2}}\right)$.

INJECTIVITY RADIUS

The following result can be proved using Corollary 122, Exercise 92.

Theorem 123 (Klingenberg's injectivity radius estimate)**.** *Let (M, g) be a compact Riemannian manifold with $K_M \leqslant C$ where $C > 0$, set*

$$l(M, g) = \int \{L(\gamma) \,|\, \gamma \text{ is a smooth closed geodesic}\}.$$

Then either $\operatorname{inj}(M) \geqslant \frac{\pi}{\sqrt{C}}$ *or* $\operatorname{inj}(M) = \frac{l(M,g)}{2}$.

5.2. Hessian and Laplacian comparisons

Theorem 124 (Hessian comparison)**.** *Let $(M, g), (\widetilde{M}, \widetilde{g})$ be two Riemannian manifolds with the same dimension , $p \in M, \widetilde{p} \in \widetilde{M}$, $U = M \setminus \operatorname{cut}(p), \widetilde{U} = \widetilde{M} \setminus \operatorname{cut}(\widetilde{p})$, $r, \widetilde{r}$ the distance from $p, \widetilde{p}$. Suppose $\gamma, \widetilde{\gamma} : [0, b] \to U, \widetilde{U}$ are two unit-speed geodesics with $\gamma(0) = p, \gamma(b) = q, \widetilde{\gamma}(0) = \widetilde{p}, \widetilde{\gamma}(b) = \widetilde{q}$. If for any t and any planes $\Sigma, \widetilde{\Sigma}$, the sectional curvatures satisfy*

$$K_\Sigma(\gamma(t)) \geqslant \widetilde{K}_\Sigma(\widetilde{\gamma}(t)),$$

then for any vectors $X \in T_q M, \widetilde{X} \in T_{\widetilde{q}} \widetilde{M}$ with $|X| = |\widetilde{X}| = 1$ and $X \perp \gamma'(b), \widetilde{X} \perp \widetilde{\gamma}'(b)$,

$$\operatorname{Hess} r(X, X) \leqslant \operatorname{Hess} \widetilde{r}(\widetilde{X}, \widetilde{X}).$$

In particular,

$$\Delta_g r|_{\gamma(t)} \leqslant \Delta_{\widetilde{g}} \widetilde{r}|_{\widetilde{g}(t)}.$$

Moreover, if the identity holds for all t, then $K_\Sigma(\gamma(t)) = \widetilde{K}_{\widetilde{\Sigma}}(\widetilde{\gamma}(t))$.

Theorem 125 (Laplacian comparison)**.** *Let (M,g) be a complete Riemannian manifold, $p \in M, U = M \backslash \operatorname{cut}(p), r$ the distance from p. If*

$$\operatorname{Ric} \geqslant (n-1)kg$$

for some constant k, then

$$\Delta_g r \leqslant (n-1)\frac{\operatorname{sn}_k'(r)}{\operatorname{sn}_k(r)}$$

on $U\backslash\{p\}$. Moreover, if the identity holds on $U\backslash\{p\}$, then (M,g) has constant sectional curvature k.

5.3. Volume comparison

Volume comparison

Write $B(p,\delta)$ for the metric ball centered at p, g_k the metric with constant sectional curvature k on $B(p,\delta)\backslash\{p\}$.

Theorem 126 (Bishop-Gromov)**.** *Let (M,g) be a complete Riemannian manifold with*

$$\operatorname{Ric} \geqslant (n-1)kg,$$

for some constant k. Then the volume ratio

$$\frac{\operatorname{Vol}_g(B(p,\delta))}{\operatorname{Vol}_{g_k}(B(p,\delta))}$$

is non-increasing for $\delta \in \mathbb{R}^+$, and

$$\lim_{\delta\to 0}\frac{\operatorname{Vol}_g(B(p,\delta))}{\operatorname{Vol}_{g_k}(B(p,\delta))} = 1.$$

Moreover, if there exists $0 < \delta_1 < \delta_2 \leqslant \delta$ with

$$\frac{\operatorname{Vol}_g(B(p,\delta_1))}{\operatorname{Vol}_{g_k}(B(p,\delta_1))} = \frac{\operatorname{Vol}_g(B(p,\delta_2))}{\operatorname{Vol}_{g_k}(B(p,\delta_2))}$$

then $\operatorname{Vol}_g(B(p,\delta)) = \operatorname{Vol}_{g_k}(B(p,\delta))$ for $\delta \in [0,\delta_2]$ and g is of constant sectional curvature on $B(p,\delta_2)$.

Theorem 127 (Zhu)**.** *Let (M,g) be a complete Riemannian manifold with*

$$\operatorname{Ric} \geqslant (n-1)kg,$$

for some constant k. Then for $0 \leqslant \delta_1 < \min\{\delta_2,\delta_3\} \leqslant \max\{\delta_2,\delta_3\} < \delta_4$,

$$\frac{\operatorname{Vol}_g(B(p,r_4)) - \operatorname{Vol}_g(B(p,r_3))}{\operatorname{Vol}_{g_k}(B(p,r_4)) - \operatorname{Vol}_{g_k}(B(p,r_3))} \leqslant \frac{\operatorname{Vol}_g(B(p,r_2)) - \operatorname{Vol}_g(B(p,r_1))}{\operatorname{Vol}_{g_k}(B(p,r_2)) - \operatorname{Vol}_{g_k}(B(p,r_1))}.$$

Proposition 128 (Gromov)**.** *Let (M,g) be a complete Riemannian manifold of dimension n with* $\operatorname{Ric} \geqslant (n-1)kg$ *for some constant $k>0$. Then*

$$\operatorname{Vol}_g(M) \leqslant \operatorname{Vol}_{g_k}\left(S^n(\frac{1}{\sqrt{k}})\right).$$

If the equality holds, then (M,g) is isometric to $S^n\left(\frac{1}{\sqrt{k}}\right)$.

Proposition 129 (Cheng)**.** *Let (M,g) be a complete Riemannian manifold of dimension n with* $\operatorname{Ric} \geqslant (n-1)kg$ *for some constant $k>0$. If* $\operatorname{diam} M = \frac{\pi}{\sqrt{k}}$*, then (M,g) is isometric to $S^n\left(\frac{1}{\sqrt{k}}\right)$.*

Combining the divergence theorem, Theorem 66, Proposition 129, we can show the following results.

Theorem 130. *Let (M,g) be a compact orientable Riemannian manifold of dimension $n \geqslant 2$. Suppose* $\operatorname{Ric} \geqslant \lambda g > 0$.

(1) (Lichnerowicz) The first non-zero eigenvalue λ_1 of the Hodge laplacian $\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$ *satisfies*

$$\lambda_1 \geqslant \frac{n}{n-1}\lambda.$$

(2) (Obata) If $\lambda_1 = \frac{n}{n-1}\lambda$, then (M,g) is isometric to the round sphere $\left(S^n\left(\sqrt{\frac{n-1}{\lambda}}\right), g_{\mathrm{can}}\right)$.

Theorem 131 (Bishop-Yau)**.** *Let (M,g) be a complete non-compact Riemannian manifold of dimension n with* $\operatorname{Ric} \geqslant 0$*. Then*

$$c_n \operatorname{Vol}_g(B(p,1))r \leqslant \operatorname{Vol}_g(B(p,r)) \leqslant \operatorname{Vol}_{g_1}(B(p,r)) = \frac{\operatorname{Vol}(S^{n-1})}{n} r^n,$$

for some positive constant c_n depending only on n and large r.

5.4. The splitting theorem

Theorem 132 (Cheeger-Gromoll)**.** *Let (M,g) be a complete Riemannian manifold of dimension n with* $\operatorname{Ric} g \geqslant 0$*. If there is a geodesic line in M, then (M,g) is isometric to $\mathbb{R}\times N, g_{\mathbb{R}} \oplus g_N$, where* $\operatorname{Ric} g_N \geqslant 0$.

Corollary 133. *Let (M,g) be a complete Riemannian manifold with* $\operatorname{Ric} \geqslant 0$.

(1) (M,g) is isometric to $(\mathbb{R}^k \times N, g_{\mathbb{R}^k} \oplus g_N)$, where N does not contain a geodesic line and $\operatorname{Ric} g_N \geqslant 0$.

(2) The isometry group splits

$$\mathrm{Isom}(M,g) \cong \mathrm{Isom}(\mathbb{R}^k, g_{\mathbb{R}^k}) \times \mathrm{Isom}(N, g_N).$$

Definition 134 (Bieberbach group)**.** *A subgroup B_n of $\mathrm{Isom}(\mathbb{R}^n, g_{\mathrm{can}}) = O(n) \rtimes \mathbb{R}^n$ is a Bieberbach group if it acts freely on $\mathbb{R}^n$ and $\mathbb{R}^n/B_n$ is a compact manifold.*

Theorem 135 (structure of manifolds with $\mathrm{Ric} \geqslant 0$)**.** *Let (M,g) be a compact Riemannian manifold with $\mathrm{Ric} \geqslant 0$, and $\pi : (\widetilde{M}, \widetilde{g}) \to (M,g)$ its universal covering with pull-back metric.*

(1) There exists some integer $k \geqslant 0$ and a compact Riemannian manifold (N, g_N) with $\mathrm{Ric}\, g_N \geqslant 0$ such that $(\widetilde{M}, \widetilde{g})$ is isometric to $(\mathbb{R}^k \times N, g_{\mathbb{R}^k} \oplus g_N)$.

(2) The isometry group splits

$$\mathrm{Isom}(M,g) \cong \mathrm{Isom}(\mathbb{R}^k, g_{\mathbb{R}^k}) \times \mathrm{Isom}(N, g_N).$$

(3) There exists a finite normal subgroup G of $\mathrm{Isom}(N,h)$, a Bieberbach group B_k and an exact sequence

$$0 \to G \to \pi_1(M) \to B_k \to 0.$$

Corollary 136. *Let (M,g) be a compact Riemannian manifold with $\mathrm{Ric} \geqslant 0$, and $\pi : (\widetilde{M}, \widetilde{g}) \to (M,g)$ its universal covering with pull-back metric.*

(1) If $\widetilde{M}$ is contractible, then $(\widetilde{M}, \widetilde{g})$ is isometric to $(\mathbb{R}^n, g_{\mathbb{R}^n})$ and (M,g) is flat.

(2) If $(\widetilde{M}, \widetilde{g})$ does not contain a line, then $\pi_1(M)$ is finite and $b_1(M) = 0$.

(3) If $\pi_1(M)$ is finite, then $\widetilde{M}$ is compact and $b_1(M) = 0$.

Corollary 137. *Let (M,g) be a compact Riemannian manifold with $\mathrm{Ric} \geqslant 0$. If there exists some point $p \in M$ such that $\mathrm{Ric}_p > 0$, then $\pi_1(M)$ is finite and $b_1(M) = 0$.*

Corollary 138. *Let (M,g) be a compact Riemannian manifold with $\mathrm{Ric} \geqslant 0$, and $\dim M = n$. Then $b_1(M) \leqslant n$. Moreover, $b_1(M) = n$ iff (M,g) is flat.*

Corollary 139. *$S^3 \times S^1$ can not admit Ricci flat metrics.*

Exercise 140. *suppose* (M^n, g) *is compact with* $b_1 = k$. *if* $\mathrm{Ric} \geqslant 0$, *show that the universal covering splits:*

$$(\widetilde{M}, g) = (N, h) \times (\mathbb{R}^k, g_{\mathbb{R}^n}).$$

give an example where $b_1 < n$ *and* $(\widetilde{M}, g) = (\mathbb{R}^n, g_{\mathbb{R}^n})$.

6. Gathering important results

(1) Koszul formula
(2) for 3-dim manifolds, Einstein implies CSC.
(3) volume expression of the Laplacian {see 10}
(4) symmetry and orthogonality of the 2nd fundamental form
(5) Gauss' lemma {see 24}
(6) Hopf-Rinow theorem {see 30}
(7) local expansion of metric {see 33}
(8) properties of the radial vector field and corollaries {see 36}
(9) expression of d* {see 44}
(10) divergence theorem {see 1.8}
(11) Ricci identity {see 57}
(12) 2nd Bianchi identity {see 60}
(13) Schur's lemma {see 61}
(14) Bochner formula for smooth functions {see 66}
(15) Bochner formula for Killing vector fields{see 68}
(16) Bochner formula for harmonic 1-forms {see 73}
(17) *Bochner formula for smooth maps {see 75}
(18) 1st and 2nd variation of the energy
(19) characterization of the Jacobi field {see 84}
(20) index theorem and topology
(21) Cartan-Hadamard theorem {see 93}

(22) characterization of CH manifolds {see 95}

(23) Cartan's fixed point and torsion free theorem {see 97, 98}

(24) Preissmann theorem {see 101}

(25) Byers theorem {see 103}

(26) no product manifold admits a metric of negative sectional curvature

(27) Myers theorem {see 104}

(28) Synge theorem {see 106}

(29) Weinstein-Synge theorem {see 108}

(30) Riemann-Hopf-Killing theorem {see 110}

(31) properties of space of CSC

(32) Rauch comparison and corollaries

(33) Hessian and Laplacian comparisons

(34) volume comparison

(35) proof of Cheng's rigidity theorem

(36) Lichnerowicz-Obata eigenvalue inequality and rigidity

(37) Cheeger-Gromoll splitting theorem and corollaries

(38) structure of manifolds with Ric $\geqslant 0$.

A. Isometry and local isometry

Definition 141 ((local) isometry). *Let $\varphi : (M, g_M) \to (N, g_N)$ be smooth.*

(1) φ is called a local isometry if $\varphi_{*,p} : T_pM \to T_{\varphi(p)}M$ is a linear isometry for every $p \in M$, or equivalently, $g_M = \varphi^* g_N$.

(2) φ is called an isometry if φ is surjective and preserve the distance.

List of properties:

- if φ is a local isometry, then φ is totally geodesic;
- for smooth curve $\gamma : [a, b] \to M$ and $\widetilde{\gamma} = \varphi \circ \gamma$, γ is a geodesic iff $\widetilde{\gamma}$ is a geodesic.

Theorem 142. *Let $\varphi : (M, g_M) \to (N, g_N)$ be smooth and bijective. The followings are equivalent*

(1) φ is an isometry.

(2) φ is a diffeomorphism and a local isometry.

(3) φ is a diffeomorphism and for every smooth curve $\gamma : [a, b] \to M$,

$$\text{length}(\varphi \circ \gamma) = \text{length}(\gamma).$$

Exercise 143. *prove the theorem above.*

B. Covering maps and transformations

Riemannian covering maps

Definition 144 (Riemannian covering map)**.** *A smooth covering map $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ is a Riemannian covering map if it is a local isometry.*

Theorem 145. *Suppose $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ is a local isometry.*

(1) If $(\widetilde{M}, \widetilde{g})$ is complete, then π is a Riemannian covering map and (M, g) is complete.

(2) If π is a covering map, then (M, g) is complete iff $(\widetilde{M}, \widetilde{g})$ is complete.

Deck transformations

Definition 146 (deck transformation)**.** *Let $\pi : \widetilde{M} \to M$ be the universal covering of M. A deck transformation $F : \widetilde{M} \to \widetilde{M}$ is a homeomorphism such that $\pi \circ F = F$, enote by $\mathrm{Aut}_\pi(\widetilde{M})$ the set of deck transformations*

Theorem 147. *(1) $\pi_1(M) \cong \mathrm{Aut}_\pi(\widetilde{M})$;*

(2) $\mathrm{Aut}_\pi(\widetilde{M})$ acts smoothly freely and properly on $\widetilde{M}$;

(3) $\mathrm{Aut}_\pi(\widetilde{M})$ acts transitively on each fiber of π.

C. Axes, rays and lines

Free homotopy class

Definition 148. *Two loops $\gamma_0, \gamma_1; [0, 1] \to M$ are said to be freely homotopic if they are homotopic through closed paths, i.e. there exists a homotopy $H(s, t) : [0, 1] \times [0, 1] \to M$ such that*

$$H(0, t) = \gamma_0(t), H(1, t) = \gamma_1(t) \text{ and } H(s, 0) = h\mathcal{H}(s, 1).$$

Axes

Definition 149 (axis of an isometry)**.** *Let (M, g) be complete, $F : M \to M$ be an isometry. A geodesic $\mathbb{R} \to M$ is called an axis of F if $F \circ \gamma$ is a non-trivial translation of γ, i.e.*

$$F(\gamma(t)) = \gamma(t + c)$$

for some constant $c \neq 0$. F is axial if it has an axis.

Lemma 150. *Let (M, g) be complete, F be an isometry. If $\delta_F(p) = d(p, F(p))$ has a positive minimum, then F has an axis.*

Theorem 151. *Let (M, g) be a compact Riemannian manifold, $F : \widetilde{M} \to \widetilde{M}$ be a non-trivial deck transformation of $\pi : \widetilde{M} \to M$.*

(1) δ_F has a positive minumum and $\delta_F \geq 2\,\mathrm{inj}(M)$, thus F is axial.

(2) The axis corresponding to this minimum is mapped under π to a closed geodesic, whose length is minimal in its free homotopy class.

Exercise 152. *suppose (M, g) is a compact connected riemannian manifold. every non-trivial free homotopy class in M is represented by a closed geodesic that has minimum length among all admissible loops in the given free homotopy class.*

Geodesic rays

Definition 153 (geodesic ray)**.** *A geodesic ray is a unit-speed geodesic $\gamma : [0, \infty) \to M$ such that $d(\gamma(s), \gamma(t)) = |s - t|$ for any $s, t \geqslant 0$.*

Lemma 154. *Let (M, g) be a complete Riemannian manifold. The followings are equivalent.*

(1) M is non-compact.

(2) For any $p \in M$, there is a geodesic ray starting from p.

Proposition 155 (definition of Busemann function)**.** *Let (M, g) be a complete Riemannian manifold, $\gamma : [0, \infty) \to M$ be a geodesic ray starting from a point p. Define*

$$b_\gamma^t(x) = d(x, \gamma(t)) - t = d(x, \gamma(t)) - d(\gamma(0), \gamma(t))$$

then $b_\gamma^t(x)$ is non-increasing for t. Define the Busemann function by

$$b_\gamma(x) = \lim_{t \to \infty} b_\gamma^t(x).$$

List of properties:

- $|b^t_\gamma(x)| \leqslant d(x, \gamma(0))$;
- $|b^t_\gamma(x) - b^t_\gamma(y)| \leqslant d(x, y)$.

Exercise 156. *compute the busemann functions on the upper half plane* $\mathbb{H}^2$ *with canonical metric of constant sectional curvature* -1.

Geodesic lines

Definition 157 (geodesic line)**.** *A geodesic line is a unit-speed geodesic* $\gamma : \mathbb{R} \to M$ *such that* $d(\gamma(s), \gamma(t)) = |s - t|$ *for any* $s, t \in \mathbb{R}$.

Lemma 158. *Let* (M, g) *be a connected complete non-compact manifold. If* M *contains a compact subset* K *such that* $M \backslash K$ *has at least two un-bounded components, then there is a geodesic passing through* K.

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