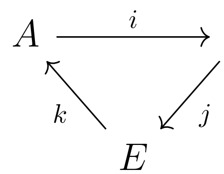
Algebraic topology 2: HW collection

2025 spring

1. Algebraic topology 2: HW1

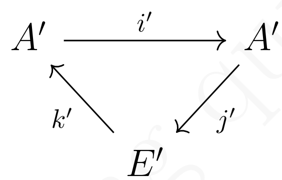
Problem 1. Check that the derived couple of an exact couple is still an exact couple.

Solution. Consider the following exact couple, where $d = j \circ k : E \to E$



The derived couple is defined by setting

- $E' = \ker d / \operatorname{im} d$
- $A' = \operatorname{im} i \subset A$
- $i' = i|_{A'}$
- $j' : i(a) \mapsto [j(a)]$
- $k' : [e] \mapsto k(e)$
- $d = j' \circ k'$



(1) First, j' and k' are well-defined.

- if $i(a) = i(b)$, then $(a - b) \in \ker i = \operatorname{im} k$, so $j(a - b) \in \operatorname{im} d$, i.e. $[j(a)] = [j(b)]$;
- if $[e] = [f]$, then $(e - f) \in \operatorname{im} j \circ k$, since $k \circ j = 0$, $k(e - f) = 0$, i.e. $k(e) = k(f)$.

(2) Second, exactness at (left) A'.
Obviously, $\operatorname{im} k' = k(\ker d)$. So $a \in \ker i' \iff a \in \ker i \cap \operatorname{im} i \iff a \in \operatorname{im} k \cap \ker j \iff a = k(b)$ with $j \circ k(b) = 0 \iff a \in \operatorname{im} k'$.

(3) Third, exactness at (right) A'.
$a = i(b) \in \ker j' \iff j(b) \in \operatorname{im} d \iff j(b) = j \circ k(c) \iff b - k(c) \in \ker j = \operatorname{im} i \iff b = k(c) + i(f) \iff a = i(b) = i \circ k(c) + i \circ i(f) = i \circ i(f) \in \operatorname{im} i'$.

(4) Forth, exactness at E'.
$[e] \in \ker k', e \in \ker d \iff e \in \ker d \cap \operatorname{im} j \iff e = j(a) \iff [e] = [j(a)] = j'(i(a)) \in \operatorname{im} j'$.

Thus the derived couple is also exact.

Problem 2. Let $C = \oplus_n C_n$ be a filtered chain complex. In other words, we have a sequence of inclusions:

$$\cdots F_pC \subset F_{p+1}C \subset \cdots$$

where each $F_pC = \oplus_n F_pC_n$ is a subcomplex of C and $\cup_p F_pC = C$. The associated graded complex is

$$GrC = \bigoplus_p Gr_pC = \bigoplus_{n,p} \frac{F_pC_n}{F_{p-1}C_n}.$$

(1) Let $A = \oplus_{n,p} F_pC_n$. Show that A and GrC form an exact couple;

(2) Prove that there is a spectral sequence with $E^1_{p,q} = H_{p+q}(Gr_pC)$;

(3) Suppose that the filtration is finite, i.e. $F_pC = F_{p+1}C$ for all but finitely many p's. Prove that the spectral sequence above converges to $E^\infty_{p,q} \cong Gr_pH_{p+q}(C)$ for some filtration on $H_{p+q}(C)$.

Solution. *(1)* We have the following short exact sequence

$$0 \to F_{p-1}C_n \hookrightarrow F_pC_n \twoheadrightarrow F_pC_n/F_{p-1}C_n \to 0,$$

from which we get a long exact sequence of homology groups:

$$\begin{array}{ccccc} \to H_n(F_{p-1}C) & \xrightarrow{i} & H_n(F_pC) & \xrightarrow{j} & H_n(F_pC/F_{p-1}C) \\ & & & & \big\downarrow k \\ & & & & H_{n-1}(F_{p-1}C) \to \end{array}$$

where i, j are induced by inclusion and quotient, and k is induced by $\partial : C_n \to C_{n-1}$. Let $M = \oplus_{n,p} H_n(F_pC)$, $E = \oplus_{n,p} H_n(F_pC/F_{p-1}C)$,

$$\begin{array}{ccc} M & \xrightarrow{\quad i \quad} & M \\ & {}_{k}\nwarrow \quad \swarrow_{j} & \\ & E & \end{array}$$

The exactness of the couple comes from the long exact sequence above.

(2) From the exact couple in *(1)*, we get a spectral sequence by taking derived couple. Write $n = p + q$, the first page consists of

$$E^1_{p,q} = H_n(F_pC/F_{p-1}C) = H_{p+q}(Gr_pC).$$

(3) In *(1)*, $M_{n,p} = H_n(F_pC)$. If the filtration is finite, then $M_{n,-\infty} = H_n(\emptyset) = 0, M_{n,\infty} = H_n(C)$. In the r-th stage, we have

$$\begin{array}{ccccccc}
E^r_{n+1,p+r-1} & & & & & & \\
\downarrow{\scriptstyle k_r} & & & & & & \\
M^r_{n,p+r-2} & \xrightarrow{i_r} & M^r_{n,p+r-1} & \xrightarrow{j_r} & E^r_{n,p} & & \\
 & & & & \downarrow{\scriptstyle k_r} & & \\
 & & & & M^r_{n-1,p-1} & \xrightarrow{j_r} & M^r_{n-1,p}
\end{array}$$

As $r \to \infty$, $M^r_{n-1,p-1}, M^r_{n-1,p}$ tend to the images of $M_{n,-\infty}$, i.e. 0. Also $E^r_{n+1,p+r-1} = H_{n+1}(Gr_{p+r-1}C)$ tends to 0. Thus

$$0 \longrightarrow M^r_{n,p+r-2} \xrightarrow{i_r} M^r_{n,p+r-1} \xrightarrow{j_r} E^r_{n,p} \xrightarrow{k_r} 0$$

tells us $E^r_{n,p} = i^{r-1}(M_{n,p})/i^r(M_{n,p-1})$. We set $i^r(M_{n,p-1}) \to F^{p-1}_n$, $i^{r-1}(M_{n,p}) \to F^p_n$, as $r \to \infty$, where $F^{p-1}_n \subset F^p_n \subset M_{n,\infty} = H_n(C)$. Write $n = p+q$, and take a filtration

$$\cdots F^{p-1}_{p+q} \subset F^p_{p+q} \subset \cdots \subset F^{\infty}_{p+q} = H_{p+q}(C),$$

then $E^{\infty}_{p,q} = Gr_pH_{p+q}(C)$.

Problem 1. Let $F \xrightarrow{i} X \xrightarrow{\pi} B$ be a fibration. Write down a commutative diagram similar to the following

$$\begin{array}{ccc} H^n(B) & \xrightarrow{\pi^*} & H^n(X) \\ \cong\big\downarrow & & \big\uparrow \\ E_2^{n,0} & \twoheadrightarrow & E_\infty^{n,0} \end{array}$$

for $H^n(F), H^n(X), E_2^{0,n}, E_\infty^{0,n}$. Label all the maps depending whether they are injective or surjective or isomorphism or i^*.

Solution. The diagram is

$$\begin{array}{ccc} H^n(X) & \xrightarrow{i^*} & H^n(F) \\ \big\downarrow\!\!\downarrow & & \big\downarrow\cong \\ E_\infty^{0,n} & \hookrightarrow & E_2^{0,n} \end{array}$$

Problem 2.[five-term exact sequence] Let $F \xrightarrow{i} X \xrightarrow{\pi} B$ be a fiber bundle over a path-connected CW complex B with trivial monodromy. Prove that there is an exact sequence:

$$0 \to H^1(B) \to H^1(X) \to H^1(F) \to H^2(B) \to H^2(X)$$

Solution. Consider the following diagram

$$\begin{array}{ccccccccc} 0 & & E_\infty^{0,1} & \hookrightarrow & E_2^{0,1} & & E_2^{2,0} & \twoheadrightarrow & E_\infty^{2,0} \\ \big\downarrow & & \big\uparrow\!\!\uparrow & & \big\uparrow\cong & & \cong\big\downarrow & & \big\downarrow \\ H^1(B) & \longrightarrow & H^1(X) & \longrightarrow & H^1(F) & \longrightarrow & H^2(B) & \longrightarrow & H^2(X) \\ \cong\big\uparrow & & \big\uparrow & & \| & & \| & & \\ E_2^{1,0} & \twoheadrightarrow & E_\infty^{1,0} & & E_2^{0,1} & \xrightarrow{\mathrm{d}_2} & E_2^{2,0} & & \end{array}$$

(1) Exactness at $H^2(B)$. Note that $E_\infty^{2,0} = E_3^{2,0}$, so the kernel is $\ker(E_2^{2,0} \twoheadrightarrow E_3^{2,0} = E_2^{2,0}/\operatorname{im}\mathrm{d}_2) = \operatorname{im}\mathrm{d}_2$;

(2) Exactness at $H^1(F)$. Note that $E_\infty^{0,1} = E_3^{0,1}$, so the image is $\operatorname{im}(E_\infty^{0,1} \to E_2^{0,1}) = E_3^{0,1} = \ker\mathrm{d}_2$;

(3) Exactness at $H^1(X), H^1(B)$. Note that $E_2^{2,0} = E_\infty^{2,0} = H^1(B)$. Thus it is exact at $H^1(B)$. Take a filtration of $H^1(X)$

$$H^1(X) = F_0 \supset F_1 = H^1(B) \supset F_2 \supset \cdots$$

The kernel at $H^1(X)$ is $\ker(H^1(X) \twoheadrightarrow E_\infty^{0,1} = F_0/F_1) = H^1(B)$.

Thus we have a five-term exact sequence.

Problem 3.[Gysin sequence] Let $S^n \to X \to B$ be a fibration where the fiber is a sphere. Suppose that the monodromy is trivial. Prove that there is an element $e \in H^{n+1}(B)$ such that the cup product with e gives a group homomorphism which fits into a long exact sequence as the following:

$$\cdots \to H^k(B) \to H^k(X) \to H^{k-n}(B) \to H^{k+1}(B) \to \cdots$$

Solution. Consider the following diagram

$$\begin{array}{ccccccc}
H^{k-n-1}(B) & & Gr_{k-n}H^k(X) & = & E_\infty^{k-n,n} & & \\
\downarrow{\scriptstyle \mathrm{d}_{n+1}} & & \twoheaduparrow & & \downarrow & & \\
H^k(B) & \xrightarrow{\pi^*} & H^k(X) & \longrightarrow & H^{k-n}(B) & \xrightarrow{\mathrm{d}_{n+1}} & H^{k+1}(B) \\
{\scriptstyle \cong}\uparrow & & \uparrow & & & & \\
E_2^{k,0} & \twoheadrightarrow & E_\infty^{k,0} & & & &
\end{array}$$

Note that $E_{n+1}^{p,q} = H^p(B)$ for $q = 0, n$ and vanishes otherwise. The map $\mathrm{d}_{n+1} : E_{n+1}^{p,q} \to E_{n+1}^{p+n+1,q-n}$ above is defined via the cup product with some $e \in H^{n+1}(B)$.

(1) Exactness at $H^{k-n}(B)$. The image is $E_\infty^{k-n,n} = E_{n+2}^{k-n,n} = \ker \mathrm{d}_{n+1}$.

(2) Exactness at $H^k(X)$. Note that $E_\infty^{p,k-p} = Gr_p H^k(X)$ is possibly non-zero only for

- $k - p = 0$, i.e. $Gr_k H^k(X)$;
- $k - p = n$, i.e. $Gr_{k-n} H^k(X)$.

Thus $H^k(X) = \oplus Gr_r H^k(X) = E_\infty^{k-n,n} \oplus E_\infty^{k,0}$, that is it.

(3) Exactness at $H^k(B)$. The kernel is $\ker(E_2^{k,0} \twoheadrightarrow E_\infty^{k,0}) = \operatorname{im} \mathrm{d}_{n+1}$.

Thus we have the required exact sequence.

Problem 4.[Wang sequence] Let $F \to X \to S^n$ be a fibration where the basis is a sphere with $n \geqslant 2$. Prove that $H^*(F)$ and $H^*(X)$ fit in a long exact sequence.

Solution. Consider the following diagram

$$
\begin{array}{ccccccc}
H^{k-1}(F) & & E_\infty^{0,k} & \hookrightarrow & E_2^{0,k} & & \\
\downarrow {\scriptstyle \mathrm{d}_{n+1}} & & \twoheaduparrow & & \uparrow \cong & & \\
H^{k-n}(F) & \longrightarrow & H^k(X) & \xrightarrow{i^*} & H^k(F) & \xrightarrow{\mathrm{d}_n} & H^{k-n+1}(F) \\
\twoheaddownarrow & & \hookuparrow & & & & \\
E_\infty^{n,k-n} & = & Gr_n H^k(X) & & & &
\end{array}
$$

(1) Exactness at $H^k(F)$. The image is $E_\infty^{0,k} = \ker \mathrm{d}_n$.

(2) Exactness at $H^k(X)$. Note that $E_\infty^{p,k-p} = Gr_p H^k(X)$ is possibly non-zero only for

- $p = 0$, i.e. $Gr_0 H^k(X)$;
- $p = n$, i.e. $Gr_n H^k(X)$.

Thus $H^k(X) = \oplus Gr_r H^k(X) = E_\infty^{n,k-n} \oplus E_\infty^{0,k}$, that is it.

(3) Exactness at $H^{k-n}(F)$. The kernel is $\ker(H^{k-n}(F) \twoheadrightarrow E_\infty^{n,k-n}) = \operatorname{im} \mathrm{d}_n$.

In conclusion, we have a long exact sequence

$$\cdots \to H^{k-1}(F) \to H^{k-n}(F) \to H^k(X) \to H^k(F) \to \cdots$$

Problem 5.[Leray-Hirsch] Let $F \xrightarrow{i} X \xrightarrow{\pi} B$ be a fiber bundle over a path-connection CW complex B. Prove that if $i^* : H^*(X;\mathbb{Q}) \to H^*(F;\mathbb{Q})$ is surjective, then we have an isomorphism of graded abelian groups

$$H^*(B;\mathbb{Q}) \otimes H^*(F;\mathbb{Q}) \cong H^*(X;\mathbb{Q}).$$

Solution. First, consider the edge morphism

$$
\begin{array}{ccc}
H^q(E) & \xrightarrow{\quad i^* \quad} & H^q(F) \\
\twoheaddownarrow & & \hookuparrow \\
E_\infty^{0,q} & \hookrightarrow & E_2^{0,q} = H^0(B; H^q(F)) = H^q(F)^{\pi_1(B)}
\end{array}
$$

so $H^q(F)^{\pi_1(B)} = H^q(F)$, which means the action of $\pi_1(B)$ on $H^*(F;\mathbb{Q})$ is trivial. For simplicity, we drop the coefficient $\mathbb{Q}$, obviously $E_2^{p,q} = H^p(B;H^1(F)) = H^p(B)\otimes H^q(F)$. Since i^* is always surjective, we have

$$\begin{array}{ccc} H^n(X) & \xrightarrow{i^*} & H^n(F) \\ \downarrow\!\!\downarrow & & \downarrow \cong \\ E_\infty^{0,n} & \overset{\cong}{\hookrightarrow} & E_2^{0,n} \end{array}$$

i.e. $\mathrm{d}_r : E_r^{0,n} \to E_r^{r,n-r+1}$ is always trivial. Note that $\mathrm{d}_2 : E_2^{p,q} = E_2^{p,0}\otimes E_2^{0,q} \to E_2^{p+2,q-1}$ is given by $\mathrm{d}_2 : E^{p,0} \to E_2^{p+2,-1}, E_2^{0,q} \to E_2^{2,q-1}$, which are both trivial, so d_2 is always trivial. Follow the same procedure, we can show that d_r is trivial $\forall r$, thus $E_\infty^{p,q} = E_2^{p,q}$. As a result, $H^n(X) = \oplus_r Gr_r H^n(X) = \oplus_{p+q=n} H^p(B)\otimes H^q(F)$.

Problem 1. Consider the following configuration space of n distinct points in $\mathbb{C}$:

$$\mathrm{Conf}_n\mathbb{C} := \{(x_1, \cdots, x_n) \in \mathbb{C}^n \mid x_i \neq x_j, \forall i \neq j\}.$$

(1) Constuct a fiber bundle $\pi : \mathrm{Conf}_{n+1}\mathbb{C} \to \mathrm{Conf}_n\mathbb{C}$. What is the fiber?

(2) Prove that $\mathrm{Conf}_n\mathbb{C}$ is an Eilenberg-Maclane space, i.e. a $K(G, 1)$.

(3) Prove that the fiber bundle π has trivial monodromy.

(4) Prove that the fiber bundle π has a continuous section, i.e. a continuous map $s : \mathrm{Conf}_n\mathbb{C} \to \mathrm{Conf}_{n+1}\mathbb{C}$ such that $\pi \circ s = \mathrm{id}$.

(5) Prove that the coholomogical Serre spectral sequence of the bundle π satisfies $E_2 = E_\infty$.

(6) Prove that $E_\infty^{p,q}$ is a free abelian group for all p, q.

(7) Prove that the extension problem is trivial and we have isomorphisms of groups

$$H^k(\mathrm{Conf}_n\mathbb{C}) \cong \bigoplus_{p=0}^{k} E_\infty^{p,k-p}.$$

Conclude that $H^k(\mathrm{Conf}_n\mathbb{C})$ is torsion free for all k.

(8) Compute Poincaré polynomial of $\mathrm{Conf}_n\mathbb{C}$.

Solution. *(1)* Define $\pi : \mathrm{Conf}_{n+1}\mathbb{C} \to \mathrm{Conf}_n\mathbb{C}$,

$$(x_1, \cdots, x_n, x_{n+1}) \mapsto (x_1, \cdots, x_n)$$

this is a fiber bundle, with fiber $\pi^{-1}(x_1, \cdots, x_n) = \mathbb{C}\backslash\{x_1, \cdots, x_n\}$.

(2) From the fibration, we get a long exact sequence of homotopy groups

$$\cdots \to \pi_k(\mathbb{C}\backslash\{n \text{ pts}\}) \to \pi_k(\mathrm{Conf}_{n+1}\mathbb{C}) \to \pi_k(\mathrm{Conf}_n\mathbb{C}) \to \cdots$$

Since $\mathbb{C}\backslash\{n \text{ pts}\} \simeq \vee_n S^1$, we have $\pi_k(\mathrm{Conf}_{n+1}\mathbb{C}) = \pi_k(\mathrm{Conf}_n\mathbb{C})$ for $k \geqslant 3$, and

$$\begin{array}{llll} 0 \to \pi_2(\mathrm{Conf}_{n+1}\mathbb{C}) \to & \pi_2(\mathrm{Conf}_n\mathbb{C}) & & \\ & \downarrow & & \\ & \mathbb{Z}^{*n} \to \pi_1(\mathrm{Conf}_{n+1}\mathbb{C}) \to \pi_1(\mathrm{Conf}_n\mathbb{C}) \to 0 & & \end{array}$$

But $\mathrm{Conf}_1\mathbb{C} = \mathbb{C}$, so π_k vanishes for $k \geqslant 2$ by induction, and only $\pi_1(\mathrm{Conf}_n\mathbb{C})$ may be non-trivial.

(3) (the background(from Arnold's paper) is the action of braid goup on punctured space, which does not permute these punctured points...) The elements in $H^1(\mathbb{C}\backslash\{n \text{ pts}\})$ are represented by the winding numbers around each punctured point, and the action of $g \in \pi_1(\mathrm{Conf}_n\mathbb{C})$ does not permute these n points (in fact, it is realized as a perturbation of these points), so it does not change the winding numbers.

(4) We only need to construct a continuous function $f : \mathbb{C}^n \to \mathbb{C}$ such that $f(x_1, \cdots, x_n) \neq x_i, \forall i$, then $s : (x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_n, f(x_i))$ is a continous section. For example, take $f = |x_1| + \cdots + |x_n| + 1$.

(5) From *(3)*, we know $E_2^{p,q} = H^p(\mathrm{Conf}_n\mathbb{C}; H^q(\mathbb{C}\backslash\{n \text{ pts}\}))$ is the tensor product of $H^p(\mathrm{Conf}_n\mathbb{C})$ and $H^q(\mathbb{C}\backslash\{n \text{ pts}\})$, the problem reduces to $H^q(\mathbb{C}\backslash\{n \text{ pts}\})$. But $H^q(\mathbb{C}\backslash\{n \text{ pts}\}) = 0$ for $q \geqslant 2$, so we only need to show $\mathrm{d}_2 : H^1(\mathbb{C}\backslash\{n \text{ pts}\}) \to H^2(\mathrm{Conf}_n\mathbb{C})$ is 0. Recall the five term exact sequence in **HW 2.**

$$\to H^1(\mathbb{C}\backslash\{n \text{ pts}\}) \xrightarrow{\mathrm{d}_2} H^2(\mathrm{Conf}_n\mathbb{C}) \xrightarrow{\pi^*} H^2(\mathrm{Conf}_{n+1}\mathbb{C})$$

as in *(4)*, we have

$$\mathrm{id} : H^2(\mathrm{Conf}_n\mathbb{C}) \xrightarrow{\pi^*} H^2(\mathrm{Conf}_{n+1}\mathbb{C}) \xrightarrow{s^*} H^2(\mathrm{Conf}_n\mathbb{C})$$

so π^* must be injective, thus $\mathrm{d}_2 = 0$. As a result, all the arrows in E_2^* are zero, thus $E_3^* = E_2^*$, similarly, we know $E_\infty^* = \cdots = E_2^*$.

(6) $E_2^{p,q} = H^p(\mathrm{Conf}_n\mathbb{C}) \otimes H^q(\mathbb{C}\backslash\{n \text{ pts}\}) = \begin{cases} (H^p(\mathrm{Conf}_n\mathbb{C}))^n & , q = 1 \\ H^p(\mathrm{Conf}_n\mathbb{C}) & , q = 0, \\ 0 & , q \geqslant 2 \end{cases}$

For $n = 1$, this is obviously true. And if $E_2^{p,q}$ is a free abelian group for $n = k$, then for $n = k + 1$, the free-ness of $H^p(\mathrm{Conf}_k\mathbb{C})$ means that there is no extension problem, so

$$H^p(\mathrm{Conf}_{k+1}\mathbb{C}) = H^p(\mathrm{Conf}_k\mathbb{C}) \oplus (H^{p-1}(\mathrm{Conf}_k\mathbb{C}))^k$$

is also a free abelian group.

(7) similar to *(6)*, all the terms $E_2^{p,q}$ are free, so there is no extension problem, we have $H^k(\mathrm{Conf}_n\mathbb{C}) \cong \oplus_{p=0}^k E_\infty^{p,k-p}$.

(8) Write $\beta_p^n = \mathrm{rank}\, H^p(\mathrm{Conf}_n\mathbb{C})$, then we have an inductive relation

$$\beta_p^{n+1} = \beta_p^n + n\beta_{p-1}^n.$$

For $n=1$, $\beta_0^1=1, \beta_k^1=0, k\geqslant 2$, so $P_1(t)=1$. Then we have

$$P_2(t)=1+t, P_3(t)=1+3t+2t^2=(1+t)(1+2t).$$

If $P_k(t)=(1+t)\cdots(1+(k-1)t)=a_{k-1}t^{k-1}+\cdots+a_0$, then

$$\begin{aligned}P_{k+1}(t)=&ka_{k-1}t^k+(a_{k-1}+ka_{k-2})t^{k-1}+\cdots+(a_1+ka_0)t+a_0\\=&(1+kt)(a_{k-1}t^{k-1}+\cdots+a_0)\\=&(1+t)\cdots(1+kt)\end{aligned}$$

Thus we have $P_n(t)=(1+t)\cdots(1+(n-1)t)$ for all n.

Problem 2. Compute the cohomology ring $H^*(K(\mathbb{Z},n);\mathbb{Q})$ for $n\geqslant 2$.

Solution. Write $B_n=K(\mathbb{Z},n)$, then we have a fibration, $\Omega B_n\to X\to B_n$, where ΩB_n is the loopspace, X is the path space, which is contractible. Recall that $\pi_k(\Omega B_n)=\pi_{k+1}(B_n)$, so $\Omega B_n=B_{n-1}$. We prove by induction that $H^*(B_n;\mathbb{Q})=\begin{cases}\mathbb{Q}[x_n] & ,n\text{ is even}\\ \mathbb{Q}[x_n]/(x_n^2) & ,n\text{ is odd}\end{cases}$, where $\deg x_n=n$.

For $n=1$, $B_1=S^1$, thus the statement is true.

Suppose it holds for $k<n$, then since there is no torsion term, $E_2^{p,q}=H^p(B_n;H^q(B_{n-1};\mathbb{Q}))=H^p(B_n;\mathbb{Q})\otimes_{\mathbb{Q}}H^q(B_{n-1};\mathbb{Q})$. But $H^{p+q}(X)$ is trivial, so $\mathrm{d}_n: E_n^{0,n-1}\to E_n^{n,0}$ is an isomorphism. We have $E_n^{0,n-1}=H^{n-1}(B_{n-1};\mathbb{Q})\cong\mathbb{Q}$ with generator x_{n-1}, $E_n^{n,0}=E_2^{n,0}=H^n(B_n;\mathbb{Q})\cong\mathbb{Q}$ with generator x_n. So up to a rescaling, we can assume $\mathrm{d}_nx_{n-1}=x_n$.

- If n is even, by hypothesis, $H^q(B_{n-1};\mathbb{Q})$ is non-zero only for $q=0,n-1$, so is $E_2^{p,q}$. Also $H^p(B_n;\mathbb{Q})=0$ for $1\leqslant p\leqslant n-1$, as a result, if $n\nmid p$, then $H^p(B_n;\mathbb{Q})=0$, thus $E_2^{p,q}=0$ for $n\nmid p$. As for $n\mid p$, $H^{kn}(B_n;\mathbb{Q})\cong\cdots\cong H^n(B_n;\mathbb{Q})=\mathbb{Q}$, and the map $\mathrm{d}_n: E_n^{kn,n-1}\to E^{(k+1)n,0}, x_{n-1}x_n^k\mapsto x_n^{k+1}$ shows that $x_n\smile x_n^k=x_n^{k+1}$, thus $H^*(B_n;\mathbb{Q})=\mathbb{Q}[x_n]$;
- If n is odd, by hypothesis, $H^q(B_{n-1};\mathbb{Q})$ is non-zero for $q=k(n-1)$, so consider $\mathrm{d}_n: E_n^{p,k(n-1)}\to E_n^{p+n,(k-1)(n-1)}$. Note that

$$\begin{aligned}\mathrm{d}_n(x_{n-1}^k)&=\mathrm{d}_n(x_{n-1})x_{n-1}^{k-1}+x_{n-1}\mathrm{d}_n(x_{n-1}^{k-1})\\&=x_nx_{n-1}^{k-1}+x_{n-1}x_nx_{n-1}^{k-2}+x_{n-1}^2\mathrm{d}_n(x_{n-1}^{k-2})\\&=\cdots=kx_nx_{n-1}^{k-1}\end{aligned}$$

  so $\mathrm{d}_n: E_n^{0,k(n-1)}\to E_n^{n,(k-1)(n-1)}$ is always an isomorphism. Since $H^{p+q}(X)$ is trivial, this tells that $E_{n+1}^*=E_\infty^*$ and is all zero except for $E_{n+1}^{0,0} = \mathbb{Q}$. Thus $H^p(B_n;\mathbb{Q}) = Q$ for $p = 0, n$ and is trivial otherwise. As a result, $H^*(B_n;\mathbb{Q}) = \mathbb{Q}[x_n]/(x_n^2)$.

According to the discussions above, the statement holds from induction.

Problem 3. Determine the Serre spectral sequence for cohomology over $\mathbb{Z}$ of a fibration

$$S^2 \to \mathbb{CP}^3 \to S^4$$

Moreover, prove that the two graded algebras

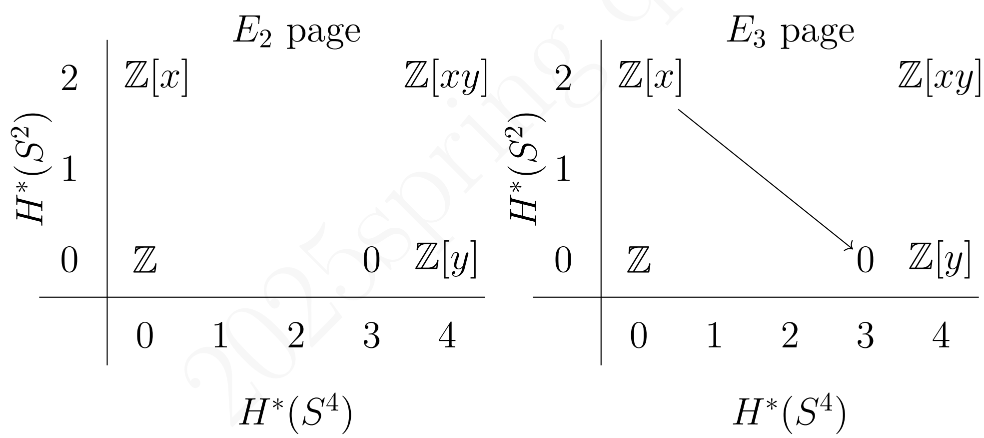
$$\bigoplus_i \bigoplus_p E_\infty^{p,i-p}$$

and

$$\bigoplus_i H^i(\mathbb{CP}^3)$$

are isomorphic as graded modules but NOT isomorphic as graded algebras.

Solution. The spectral sequence is



where $E_\infty^{p,q} = E_2^{p,q}$. Thus $E = \oplus_{i,p} E_\infty^{p,i-p} = \mathbb{Z}^4$. But the ring structure of E is $\mathbb{Z}[x,y]/(x^2,y^2) \ncong \mathbb{Z}[x]/(x^4)$. This is for the product structure of $E_2^{p,q}$ and the cup product are not compatible, the product on $E_2^{p,q}$(as a quotient) can not be lifted to $H^*(\mathbb{CP}^3)$.

Remark. This fibration is called the twistor fibration as in this paper.

We begin by recalling the *Hopf map* $\rho\colon \mathbb{C}P^3 \to \mathbb{H}P^1$. If we identify $\mathbb{C}^4$ with the left quaternionic vector space $\mathbb{H}^2$ via $(z_1, z_2, z_3, z_4) \leftrightarrow (z_1 + z_2 j, z_3 + z_4 j)$ then the Hopf map $\rho\colon \mathbb{C}P^3 \to \mathbb{H}P^1$ is given by $\rho(\mathbb{C}\mathbf{v}) = \mathbb{H}\mathbf{v}$, where $\mathbf{v} \in \mathbb{C}^4 = \mathbb{H}^2$ is nonzero.

The *twistor fibration* $\pi\colon \mathbb{C}P^3 \to S^4$ is obtained by composing ρ with the identification of $\mathbb{H}P^1$ and $S^4 \subset \mathbb{H} \oplus \mathbb{R} = \mathbb{R}^5$ given in the usual way by stereographic projection from the south pole of S^4 onto the equatorial 4-plane $\mathbb{H}$ included in $\mathbb{H}P^1$ by $q \mapsto [1, q]$. Specifically, this identification is given by

$$[q_1, q_2] \in \mathbb{H}P^1 \leftrightarrow \frac{(2\bar{q}_1 q_2, |q_1|^2 - |q_2|^2)}{|q_1|^2 + |q_2|^2} \in S^4, \tag{2.1}$$

so, writing $\mathbb{R}^5$ as $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{R}$, π is given by

$$\pi([z_1, z_2, z_3, z_4]) = \frac{(2(\bar{z}_1 z_3 + z_2 \bar{z}_4),\ 2(\bar{z}_1 z_4 - z_2 \bar{z}_3),\ |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2)}{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2}. \tag{2.2}$$

Problem 4. Compute the cohomology ring $H^*(F;\mathbb{Z})$ for F the homotopy fiber of a map $f : S^n \to S^n$ of degree k for $k, n > 1$.

Solution. Write F_n for the fiber, recall the Wang sequence in HW2

$$\cdots \to H^r(S^n) \xrightarrow{i^*} H^r(F_n) \xrightarrow{\mathrm{d}_n} H^{r+1-n}(F_n) \to H^{r+1}(S^n) \to \cdots$$

Take $r > n+1$, we have $H^r(F_n) = H^{r-(n-1)}(F_n)$, so we only need to consider $0 \leqslant r \leqslant n$. Take $0 < r < n-1$, we have $H^r(F_n) = 0$, and the left items are

$$0 \to \mathbb{Z} \to H^0(F_n) \to 0$$

$$0 \to H^{n-1}(F_n) \to H^0(F_n) \to H^n(S^n) \to H^n(F_n) \to 0$$

i.e. $H^0(F_n) = \mathbb{Z}$ and

$$0 \to H^{n-1}(F_n) \xrightarrow{\mathrm{d}_n} \mathbb{Z} \to H^n(S^n) = \mathbb{Z} \xrightarrow{i^*} H^n(F_n) \to 0$$

Recall the long exact sequence of homotopy groups

$$\cdots \to \pi_n(S^n) = \mathbb{Z} \xrightarrow{f_*} \pi_n(S^n) = \mathbb{Z} \to \pi_{n-1}(F_n) \to 0 \to \cdots$$

Since $\deg f = k$, we have $\pi_{n-1}(F_n) = \mathbb{Z}/k\mathbb{Z}$. So Hurewicz theorem tells that $H_{n-1}(F_n) = \pi_{n-1}(F_n) = \mathbb{Z}/k\mathbb{Z}$. Useing Universal coefficient theorem, $H^{n-1}(F_n) = 0, H^n(F_n) = \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(F_n), \mathbb{Z}) = \mathbb{Z}/k\mathbb{Z}$.

As a result, $H^r(F_n) = \begin{cases} \mathbb{Z}/k\mathbb{Z} & , r = n + s(n-1) \\ \mathbb{Z} & , r = 0 \\ 0 & , \text{otherwise} \end{cases}$. For the cup product structure, note that for any $x \in H^{n+s(n-1)}(F_n)$,

$$x^2 \in H^{2n+2s(n-1)}(F_n) = H^{n+1+(2s+1)(n-1)}(F_n) = 0.$$

Thus the ring structure is simply

$$\mathbb{Z}[x_n, x_{2n-1}, \cdots]/(x_n^2, kx_n, x_{2n-1}^2, kx_{2n-1}, \cdots).$$

4. Algebraic topology 2: HW4

Problem 1.

(1) Let $f : S^3 \to K(\mathbb{Z}, 3)$ be a map that induces an isomorphism on π_3. Let X be a homotopy fiber of f (assuming that it is a CW complex). Show that X is 3-connected and that $\pi_i(X) \cong \pi_i(S^3)$ for $i > 3$.

(2) Show that the fibration above gives a fibration

$$K(\mathbb{Z}, s) \to X \to S^3.$$

(3) Consider the Serre spectral sequence of the second fibration above for cohomology over integers. Compute E_2 as a graded algebra in terms of generators and relations. Show that $E_2 = E_3$.

(4) Determine the E_3-differentials. Determine E_∞.

(5) Compute $H^i(X; \mathbb{Z})$ and $H_i(X; \mathbb{Z})$ for all i.

(6) Conclude $\pi_4(S^3) \cong \mathbb{Z}/2$.

(7) Let p be a prime. Prove that the first p-torsion in $\pi_i(X)$ is a $\mathbb{Z}/p$ in π_{2p}. Conclude the same for S^3.

(8) Using the Hopf bundle, deduce the same for S^2.

(9) Using the stability of homotopy groups of spheres, show that $\pi_{n+1}(S^n) \cong \mathbb{Z}/2$ for $n \geqslant 3$.

Solution. *(1)* Consider the long exact sequence of homotopy groups

$$\to \pi_{n+1}(k(\mathbb{Z}, 3)) \to \pi_n(X) \to \pi_n(S^3) \to \pi_n(K(\mathbb{Z}, 3)) \to$$

for $n = 0, 1$, we get $\pi_n(X) = \pi_n(S^3) = 0$, for $n = 2$, we get

$$\to \pi_3(S^3) \xrightarrow{\sim} \pi_3(K(\mathbb{Z}, 3)) \to \pi_2(X) \to 0,$$

so $\pi_2(X) = 0$, for $n = 3$, we get

$$0 \to \pi_3(X) \to \pi_3(S^3) \xrightarrow{\sim} \pi_3(K(\mathbb{Z}, 3)) \to,$$

so $\pi_3(X) = 0$. And for $n \geqslant 4$, we get

$$0 \to \pi_n(X) \to \pi_n(S^3) \to 0,$$

so $\pi_n(X) = \pi_n(S^3)$.

(2) Suppose $F \to X \to S^3$ is a fibration, then we have

$$\to \pi_{n+1}(X) \to \pi_{n+1}(S^3) \to \pi_n(F) \to \pi_n(X) \to \pi_n(S^3) \to .$$

So for $n \geqslant 4$ and $n \leqslant 1$, $\pi_n(S^3) = \pi_n(X)$, thus $\pi_n(F) = 0$. Also, for $n = 2$, we get

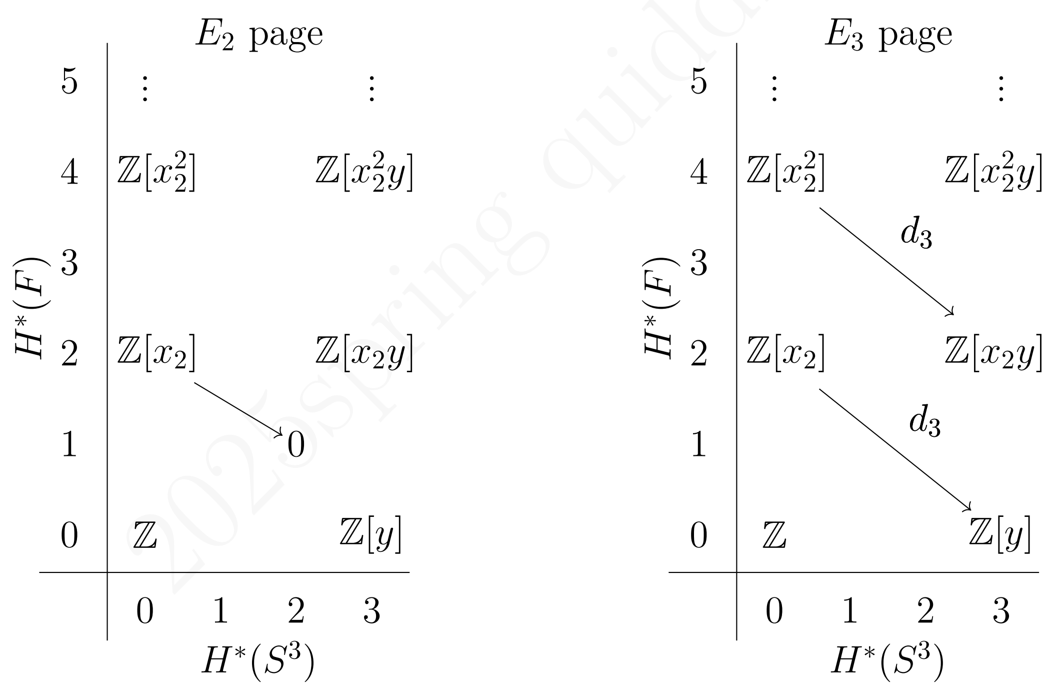
$$\to \pi_3(X) = 0 \to \pi_3(S^3) \to \pi_2(F) \to 0,$$

so $\pi_2(F) = \mathbb{Z}$, for $n = 3$, we get

$$\to \pi_4(X) \xrightarrow{\sim} \pi_4(S^3) \to \pi_3(F) \to 0,$$

so $\pi_3(F) = 0$. In conclusion F is a $K(\mathbb{Z}, 2)$ space.

(3) $E_2^{p,q} = H^p(S^3; H^q(F))$. Recall that $H^p(S^3) \neq 0$ only for $p = 0, 3$, and $H^*(K(\mathbb{Z}, 2)) = \mathbb{Z}[x_2]$, so the E_2 and E_3 pages are



So $E_2 = E_3$.

(4) Using Hurewicz theorem for X, $H^2(X) = H^3(x) = 0$, so we have from Wang sequence

$$H^2(X) = 0 \to H^2(F) \xrightarrow{\mathrm{d}_3} H^0(F) \to H^3(X) = 0,$$

which means $\mathrm{d}_3 x_2 = \pm y$, WLOG, assume $\mathrm{d}_3 x_2 = y$. Moreover,

$$\mathrm{d}_3 x_2^k = k x_2^{k-1} \mathrm{d}_3 x_2 = k x_2^{k-1} y$$

so the $E_4 = E_\infty$ page is

$H^*(F)$ \ $H^*(S^3)$	0	1	2	3
5	$\vdots$			$\vdots$
4	0			$\mathbb{Z}/3\mathbb{Z}$
3				
2	0			$\mathbb{Z}/2\mathbb{Z}$
1				
0	$\mathbb{Z}$			0

(5) $E_4^{p,q} = Gr_p H^{p+q}(X)$, from the diagram above, the only possibly non-trivial terms are $Gr_3 H^{2q+3}(X) = E_4^{3,2q} = \mathbb{Z}/(q+1)\mathbb{Z}, Gr_0 H^0(X) = \mathbb{Z}, Gr_0 H^{2q+2}(X) = 0$ for $q \geqslant 0$. Thus

$$\widetilde{H}^i(X) = \begin{cases} \mathbb{Z}/(q+1)\mathbb{Z} & , i = 2q+3, q \geqslant 0 \\ 0 & , otherwise \end{cases}.$$

Using the following exact sequence

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}(H^{q+1}(X), \mathbb{Z}) \to H_q(X) \to \operatorname{Hom}_{\mathbb{Z}}(H^q(X), \mathbb{Z}) \to 0,$$

$$\widetilde{H}_i(X) = \begin{cases} \mathbb{Z}/(q+1)\mathbb{Z} & , i = 2q+2, q \geqslant 0 \\ 0 & , otherwise \end{cases}.$$

(6) Using Hurewicz theorem again, $\pi_4(S^3) = \pi_4(X) = H_4(X) = \mathbb{Z}/2\mathbb{Z}$.

(7) It is easy to see that $\pi_i(X)$ is always a torsion group for $i > 3$. For prime p, let $\mathcal{C}_p$ be the class of torsion groups, such that any $G \in \mathcal{C}_p, g \in G$, we have $\operatorname{ord} g \mid (p!)^k$ for some $k \geqslant 1$ (i.e. no factors of any prime $q > p$). The property of $\mathcal{C}_p$ is preserved by tensor product and short exact sequence, so $\mathcal{C}_p$ is a Serre class.

Write p' for the prime next to p. Using generalized Hurewicz theorem, since $H_i(X) \in \mathcal{C}_p$, for $i < 2p'$, so $\pi_i(X) \in \mathcal{C}_p$ for $i < 2p'$. As a result, for any prime p', $\pi_{2p'}(X)$ is the first one which possibly contains some p'-torsion. Also the map

$$h : \pi_{2p'}(X) \to H_{2p'}(X) = \mathbb{Z}/p'\mathbb{Z}$$

is a $\mathcal{C}_p$ isomorphism, so the cokernel, is both a quotient of $\mathbb{Z}/p'\mathbb{Z}$ and a $(p!)^k$-torsion for some k, thus it must be 0, which means h is surjective. So there must be some p'-torsion element in $\pi_{2p'}(X)$ (e.g. $h^{-1}(1)$), then we know $\mathbb{Z}/p'\mathbb{Z}$ is a subgroup of $\pi_{2p'}(X)$.
In conclusion, $\pi_{2p}(X)$ is the first one to contains $\mathbb{Z}/p\mathbb{Z}$ as a subgroup. Then from *(1)*, this is also true for S^3.

(8) For $S^1 \to S^3 \to S^2$, we have a long exact sequence

$$\to \pi_n(S^1) \to \pi_n(S^3) \to \pi_n(S^2) \to \pi_{n-1}(S^1) \to .$$

For $n \geqslant 3$, $\pi_n(S^3) = \pi_n(S^2)$, so the result in *(7)* is also true for S^2.

(9) According to Freudenthal theorem, the map $\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$ is an isomorphism for $i < 2n-1$. For $n \geqslant 2, i = n+1$,

$$\pi_{n+1}(S^n) = \pi_n(S^{n-1}) = \cdots = \pi_4(S^3) = \mathbb{Z}/2\mathbb{Z}.$$

Problem 2. Compute $H^1(S^1;\mathbb{Z})$ with local coefficients where the action of $\pi_1(S^1)$ on $\mathbb{Z}$ is nontrivial (there is only one such action). Write down a cellular chain complex and its differentials.

Solution. The cellular structure of $\mathbb{R} = \widetilde{S^1}$ is

$$\mathbb{Z}[t,t^{-1}] \xrightarrow{t^m \mapsto t^m - t^{m-1}} \mathbb{Z}[t,t^{-1}] \to 0$$

so the cellular complex for the local system is

$$0 \to \mathrm{Hom}_{\mathbb{Z}[t,t^{-1}]}(\mathbb{Z}[t,t^{-1}],\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}[t,t^{-1}]}(\mathbb{Z}[t,t^{-1}],\mathbb{Z})$$

or $t^m \mapsto (-1)^m k$ gives $\mathrm{Hom}_{\mathbb{Z}[t,t^{-1}]}(\mathbb{Z}[t,t^{-1}],\mathbb{Z}) \cong \mathbb{Z}$, in which case

$$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z}$$

where i is given by $k \mapsto l$:

$$\begin{array}{ccc} \mathbb{Z}[t,t^{-1}] & \xrightarrow{t^m \mapsto t^m - t^{m-1}} & \mathbb{Z}[t,t^{-1}] \\ & {\scriptstyle l:t^m \mapsto (-1)^m k - (-1)^{m-1}k} \searrow \quad \swarrow {\scriptstyle k:t^m \mapsto (-1)^m k} & \\ & \mathbb{Z} & \end{array}$$

that is $(k : 1 \to k) \in \mathbb{Z} \mapsto (2k : 1 \to 2k) \in \mathbb{Z}$, so $i = 2\cdot$, and $H^1(S^1;\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

Problem 3. Let $\pi = \pi_1(S^1)$. Compute $H_*(S^1;\mathbb{Z}[\pi])$ and $H^*(S^1;\mathbb{Z}[\pi])$ from definition. Write down a chain complex and its differentials.

Solution. Since

$$\mathbb{Z}[t,t^{-1}] \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}[t,t^{-1}] = \mathrm{Hom}_{\mathbb{Z}[t,t^{-1}]}(\mathbb{Z}[t,t^{-1}], \mathbb{Z}[t,t^{-1}]) = \mathbb{Z}[t,t^{-1}]$$

the chain complexes are

$$\mathbb{Z}[t,t^{-1}] \xrightarrow{(1-t^{-1})\cdot} \mathbb{Z}[t,t^{-1}] \to 0$$

and

$$0 \to \mathbb{Z}[t,t^{-1}] \xrightarrow{(1-t^{-1})\cdot} \mathbb{Z}[t,t^{-1}].$$

So $H_0(S^1;\mathbb{Z}[\pi]) = \mathbb{Z}[t,t^{-1}]/\mathrm{im}\,((1-t^{-1})\cdot) = \mathbb{Z}, H^1(S^1;\mathbb{Z}[\pi]) = \mathbb{Z}$, the others are all 0.

Problem 4.(2-C in [MS]) Existence theorem for Euclidean metrics. Using a partition of unity, show that any vector bundle over a paracompact base space can be given a Euclidean metric.

Solution. Let $\{U_\lambda\}$ be an countable atalas of B. For each U_λ, write $\varphi_\lambda : \pi^{-1}(U_\lambda) \cong U_\lambda \times \mathbb{R}^r \cong \mathbb{R}^n \times \mathbb{R}^r$. We then define a Euclidean metric on $\pi^{-1}(U_\lambda)$, by putting the standard inner product on $\mathbb{R}^r$, i.e.

$$\langle (x,v),(x,w)\rangle = \langle v,w\rangle_{\mathbb{R}^r}, \forall x \in \mathbb{R}^n.$$

and take $\mu_\lambda((x,v)) = \langle (x,v),(x,v)\rangle$. Now take a partition of unity $\{\rho_\lambda\}$ subordinate to $\{U_\lambda\}$, let $\mu = \sum_\lambda \rho_\lambda \mu_\lambda$, this gives a Euclidean metric on the vector bundle.

Problem 5.(3-D in [MS]) If a vector bundle ξ possesses a Euclidean metric, show that ξ is isomorphic to its dual bundle $\mathrm{Hom}(\xi, \varepsilon^1)$.

Solution. Suppose μ is a Euclidean metric on ξ, then the quadratic form $\langle v,w\rangle = \frac{1}{2}\left(\mu(v+w) - \mu(v) - \mu(w)\right)$ is an inner product.
Now define a map $\varphi : E(\xi) \to E(\mathrm{Hom}(\xi,\varepsilon^1))$,

$$v \in \pi^{-1}(x) \mapsto \langle v,\cdot\rangle : \pi^{-1}(x) \to \mathbb{R}$$

First, this is a bundle map, since it preserves the base point. Second, on each fiber, according to linear algebra, the map $v \to \langle v,\cdot\rangle$ is a linear isomorphism. Thus φ ia a bundle isomorphism.

Problem 6.(3-E in [MS]) Show that the set of isomorphism classes of 1-dimensional vector bundles over B forms an abelian group with respect to the tensor product operation. Show that a given $\mathbb{R}^1$-bundle ξ possesses a Euclidean metric if and only if ξ represents an element of order $\leqslant 2$ in this group.

Solution. First, we show the group structure. For any $[\xi], [\eta]$, the product is defined by $[\xi \otimes_{\mathbb{R}} \eta]$.

- it is well-defined, since if $\xi \cong \xi', \eta \cong \eta'$, then on each fiber, as linear spaces, $\pi_\xi^{-1}(x) \cong \pi_{\xi'}^{-1}(x)$, and $\pi_\eta^{-1}(x) \cong \pi_{\eta'}^{-1}(x)$, then $\pi_{\xi\otimes\xi'}^{-1}(x) \cong \pi_{\xi'\otimes\eta'}^{-1}(x)$, so $\xi \otimes \eta \cong \xi' \otimes \eta'$;
- it is associative according to the associativity of tensor product;
- $\xi \otimes \eta \cong \eta \otimes \xi$, so the product is Abelian;
- $\xi \otimes \varepsilon^1 \cong \xi$, so $[\varepsilon^1]$ is the identity element;
- $\xi \otimes \xi^\vee \cong \varepsilon^1$(verified on each fiber with linear algebra).

In conclusion, $\{[\xi]\}$ has a structure of an Abelian group.
Second, if ξ possesses a Euclidean metric, then **Problem 5.** tells that $\xi \cong \xi^\vee$, thus $[\xi]^2 = [\xi \otimes \xi] = [\varepsilon^1]$, i.e. $[\xi]$ is of order $\leqslant 2$. Conversely, if $\xi \otimes \xi \cong \varepsilon^1$, then write $\varphi : \xi \cong \xi^\vee$. Up to a scaling, we can assume $1 \mapsto 1$ in each fiber, then $\langle \cdot, \cdot \rangle : \pi^{-1}(x) \otimes \pi^{-1}(x) \to \mathbb{R}$,

$$\langle v, w \rangle = \varphi(v)(w) = vw$$

is an inner product on $\pi^{-1}(x)$. Similar to **Problem 4.**, we can patch it to get a Euclidean metric on ξ.

5. Algebraic topology 2: HW5

Problem 1.(4-A in [MS]) Show that the Stiefel-Whitney classes of a Cartesian product are given by

$$w_k(\xi \times \eta) = \sum_{i=0}^{k} w_i(\xi) \times w_{k-i}(\eta).$$

Solution. Let p_1, p_2 be the projection of $B(\xi \times \eta)$ on to $B(\xi), B(\eta)$. Note that $E(\xi \times \eta) = p_1^* E(\xi) \oplus p_2^* E(\eta)$, so use the product formula,

$$w_k(\xi \times \eta) = \sum_{i=0}^{k} p_1^* w_i(\xi) \smile p_2^* w_{k-i}(\eta).$$

Drop the pull-back for simplicity, we get the required formula.

Problem 2.(4-B in [MS]) Prove the following theorem of Stiefel. If $n+1 = 2^r m$ with m odd, then there do not exist 2^r vector fields on the projective space $\mathbb{P}^n$, which are everywhere linearly independent.

Solution. For $m = 1$, $n = 2^r - 2 < 2^r$, so the statement is obvious. Now suppose $m > 1$, and there exist 2^r such vector fields on $\mathbb{P}^n$. In this case, we have a decomposition $T\mathbb{P}^n = \varepsilon^{2^r} \oplus \tau$, where τ is the complement bundle of rank $2^r(m-1) - 1$. Thus for $k \geqslant 2^r(m-1)$,

$$w_k(\mathbb{P}^n) = w_k(\tau) = 0.$$

But this is impossible, since take $k = 2^r(m-1)$, the number

$$\binom{2^r m}{2^r(m-1)} = \frac{(2^r(m-1)+1)\cdots 2^r m}{1 \cdots 2^r} = \prod_{k=1}^{2^r} \frac{2^r(m-1)+k}{k}$$

is an odd number, so $w_k(\mathbb{P}^n)$, as the degree k part of $(1+a)^{n+1}$, must be non-zero. This contradiction tells that there are no such vector fields.

Problem 3.(4-C in [MS]) A manifold M is said to admit a field of tangent k-planes if its tangent bundle admits a sub-bundle of dimension k. Show that $\mathbb{P}^n$ admits a field of tangent 1-planes if and only if n is odd. Show that $\mathbb{P}^4$ and $\mathbb{P}^6$ do not admit fields of tangent 2-planes.

Solution. First consider 1-planes.

- If n is even, then $w(\mathbb{P}^n) = (1+a)^{n+1} = \cdots + a^n$, i.e. $w_n(\mathbb{P}^n) = a^n \neq 0$. If there exists a 1-plane, then we have a decomposition $T\mathbb{P}^n = \xi \oplus \eta$, where ξ is of rank 1, η is of rank $n-1$. Note that $a^n = w_n(\mathbb{P}^n) = w_1(\xi)w_{n-1}(\eta)$, so $w_1(\xi) = a \neq 0$. Hence

$$(1+a)^{n+1} = w(\mathbb{P}^n) = w(\xi)w_\eta = (1+a)w(\eta),$$

i.e. $w(\eta) = (1+a)^n$, this is impossible due to rank reason.

- If n is odd, we know there is a non-vanishing vector field on S^n, thus the induced vector field on $\mathbb{P}^n$ is non-vanishing, this gives a 1-plane.

Second consider 2-planes. If there exists a 2-plane, similarly, we write $T\mathbb{P}^n = \xi \oplus \eta$, where ξ is of rank 2, η is of rank $n-2$, so $a^n = w_n(\mathbb{P}^n) = w_2(\xi)w_{n-2}(\eta)$, as a result, $w_2(\xi) = a^2 \neq 0$,

$$(1+a)^{n+1} = w(\mathbb{P}^n) = w(\xi)w(\eta) = (1+ka+a^2)w(\eta).$$

Here that $k \neq 0$, due to rank reason of $w(\eta)$, so $k = 1$.

- As for $\mathbb{P}^4$,

$$\begin{aligned} w(\eta) &= (1+a+a^2)^{-1}(1+a)^5 \\ &= (1+a+a^3+a^4)(1+a+a^4), \end{aligned}$$

there exist a^4 term, which is impossible.

- As for $\mathbb{P}^6$,

$$\begin{aligned} w(\eta) &= (1+a+a^2)^{-1}(1+a)^7 \\ &= (1+a+a^3+a^4+a^6)(1+a+\cdots+a^6), \end{aligned}$$

there exists a^6 term, which is impossible.

Problem 4.(4-D in [MS]) If the n-dimensional manifold M can be immersed in $\mathbb{R}^{n+1}$ show that each $w_i(M)$ is equal to the i-fold cup product $w_1(M)^i$. If $\mathbb{P}^n$ can be immersed in $\mathbb{R}^{n+1}$ show that n must be of the form $2^r - 1$ or $2^r - 2$.

Solution. If $i : M^n \to \mathbb{R}^{n+1}$ is an immersion, then we have a decomposition $i^*T\mathbb{R}^{n+1} = TM \oplus \gamma$, as a result

$$1 = w(M)w(\gamma) = w(M)(1+t).$$

If $t = 0$, $w(M) = 1$, and if $t \neq 0$, $w(M) = 1 + t + \cdots + t^n$, in both case, $w_i(M) = (w_1(M))^i$. Now let $M = \mathbb{P}^n$.

- if $t = 0$, we should have
$$1 = (1+a)^{n+1},$$
this happens if and only if $n+1 = 2^r$ for some r. (in **Problem 2.** we have proved $2^r(2k+1)$ is not possible for $k > 0$)
- if $t \neq 0$, we should have
$$1 = (1+a)^{n+1}(1+a),$$
similarly, this happens if and only if $n+2 = 2^r$ for some r.

Thus n must be of the form $2^r - 1$ or $2^r - 2$.

Problem 5.(4-E in [MS]) Show that the set $\mathfrak{N}_n$ consisting of all unoriented cobordism classes of smooth closed n-manifolds can be made into an additive goup. This cobordism group $\mathfrak{N}_n$ is finite by 4.11, and is clearly a module over $\mathbb{Z}/2\mathbb{Z}$. Using the manifolds $\mathbb{P}^2 \times \mathbb{P}^2$ and $\mathbb{P}^4$, show that $\mathfrak{N}_4$ contains at least four distinct elements.

Solution. The group structure

- the addition is $[M] + [N] = M \sqcup N$, this is well defined since if $M \sim M', N \sim N'$ or say $\partial A = M \sqcup M', \partial B = N \sqcup N'$, then $\partial(A \sqcup B) = (M \sqcup N) \sqcup (M' \sqcup N')$, i.e. $M \sqcup N \sim M' \sqcup N'$.
- the addition is associative and commutative, since $([M]+[N])+[L] = [M \sqcup N \sqcup L] = [M] + ([N]+[L]), [M]+[N] = [M \sqcup N] = [N]+[M]$.
- the zero element is $[\emptyset]$.
- the inverse of $[M]$ is itself, since $\partial(M \times [0,1]) = M \sqcup M$.

Thus $\mathfrak{N}_n$ is an Abelian group consisting of 2-torsions.
As for $\mathfrak{N}_4$, we compute the SW numbers, write $m_{i,j,k,l}(M)$ for the SW number of M for $w_1^i w_2^j w_3^k w_4^l$.

	$m_{0,0,0,1}$	$m_{1,0,1,0}$	$m_{0,2,0,0}$	$m_{2,1,0,0}$	$m_{4,0,0,0}$
$\emptyset$	0	0	0	0	0
$\mathbb{P}^2 \times \mathbb{P}^2$	1	0	1	0	0
$\mathbb{P}^4$	1	0	0	0	1
$(\mathbb{P}^2 \times \mathbb{P}^2) \sqcup \mathbb{P}^4$	0	0	1	0	1

($\mathbb{P}^2 \times \mathbb{P}^2$ is computed with **Problem 1.**, $(\mathbb{P}^2 \times \mathbb{P}^2) \sqcup \mathbb{P}^4$ is done by summing up)
From this table, $\emptyset, S^4, \mathbb{P}^2 \times \mathbb{P}^2, \mathbb{P}^4$ are in different cobordism classes.

Problem 6.(5-E in [MS]) Let ξ be an $\mathbb{R}^n$-bundle over B.

(1) Show that there exists a vector bundle η over B with $\xi \oplus \eta$ trivial if and only if there exists a bundle map $\xi \to \gamma^n(\mathbb{R}^{n+k})$ for large k. If such a map exists, ξ will be called a bundle of finite type.

(2) Now assume that B is normal. Show that ξ has finite type if and only if B is covered by finitely many open sets $U_1, \cdots, U_r$ with $\xi|_{U_i}$ trivial.

(3) If B is paracompact and has finite covering dimension, show (using the argument of 5.9) that every ξ over B has finite type.

(4) Using Stiefel-Whitney classes, show that the vector bundle γ^1 over $\mathbb{P}^\infty$ does not have finite type.

Solution. *(1)* For "$\Leftarrow$", since $\gamma^n(\mathbb{R}^{n+k})$ is a sub-bundle of $\varepsilon^{n+k}_{G_n(\mathbb{R}^{n+k})}$, ξ is also a sub-bundle of a trivial bundle. For "$\Rightarrow$", in order to construct a bundle map $f : \xi \to \gamma^n(\mathbb{R}^{n+k})$, it suffices to construct a linear and injective map $\widehat{f} : E(\xi) \to \mathbb{R}^{n+k}$, since the required map can be defined by

$$f(e) = (\widehat{f}(e), \widehat{f}(F_e)).$$

This is already done by $\xi \hookrightarrow \varepsilon^{n+k}_B$.

(2) "$\Leftarrow$" follows from **Lemma 5.3.** in [MS], since the proof uses compactness only for a finite r (the normal property is used here). For "$\Rightarrow$", consider the bundle map

$$\begin{array}{ccc} E(\xi) & \longrightarrow & E(\gamma^n(\mathbb{R}^{n+k})) \\ \downarrow & & \downarrow \\ B & \longrightarrow & G_n(\mathbb{R}^{n+k}) \end{array}$$

So we can pull back a trivialization covering of $G_n(\mathbb{R}^{n+k})$ to get a trivialization covering of B. The later one is compact, so the covering can be taken to be finite.

(3) Being of finite covering dimension means that, there exists some $d < \infty$, such that for any open covering, there is a refinement, in which each point is contained in no more that d open sets.

Now we mimick the proof of **Lemma 5.9.** in [MS]. Choose a locally finite open covering $\{V_\alpha\}$ such that $\sigma|_{V_\alpha}$ is trivial, and (up to a refinement) suppose each point is covered for no more than d times.

Choose an open covering $\{W_\alpha\}$ with $\overline{W}_\alpha \subset V_\alpha$. Let $\lambda_\alpha : B \to \mathbb{R}$ be a continuous function which equals 1 on $\overline{W}_\alpha$ and equals 0 outside of V_α. Now for $S \subset \{\alpha\}$, let $U(S) \subset B$ be the set of all b with

$$\min_{\alpha \in S} \lambda_\alpha(b) > \max_{\alpha \in S} \lambda_\alpha(b),$$

and let U_k be the union of $U(S)$ with $\#S = k$. So U_k is open and $B = \cup_{k=1}^{\infty} U_k$. Note that from this definition, $b \in U_k$ if and only if $\lambda_\alpha(b) > 0$ for exactly k's α, which means $b \in V_\alpha$ for k's set V_α. Thus by assumption $B = \cup_{k=1}^{d} U_k$.

Now use *(2)*, B must be of finite type.

(4) If γ^1 is of finite type, then there exists a vector bundle η over $\mathbb{P}^\infty$ with $\xi \oplus \eta$ trivial. In this case

$$1 = w(\xi \oplus \eta) = (1 + a)w(\eta)$$

so $w(\eta) = 1 + a + \cdots$, i.e. η is not of finite rank, that is impossible.

Problem 7. Consider vector bundles over a paracompact base B.

(1) Let $Vect_n(B)$ denote the set of isomorphism classes of n-dimensional real vector bundles over B. Prove that $Vect_1(B)$ forms a group under tensor product.

(2) Suppose ξ is a 1-dimensional real vector bundle over B. Prove that ξ is trivial if and only if $w_1(\xi)$ is trivial.

(3) Prove that $Vect_1(B) \cong H^1(B; \mathbb{Z}/2\mathbb{Z})$ as groups.

(4) Does *(2)* hold for $n > 1$ dimensional real vector bundles? Justify your answer.

Solution. *(1)* The group structure

- the product is $\xi \cdot \eta = \xi \otimes \eta$, it is associative and commutative, by the associativity and commutativity (up to an isomorphism) of tensor product on each fiber.
- the zero element is ε_B^1, since $\xi \otimes \varepsilon_B^1 = \xi$ for any ξ.
- the inverse of ξ is $\xi^\vee = Hom(\xi, \varepsilon_B^1)$, by the pairing.

(2) "$\Rightarrow$" is obvious. For "$\Leftarrow$", see *(3)*.

(3) The map $\varphi : Vect_1(B) \to H^1(B; \mathbb{Z}/2\mathbb{Z}), \xi \mapsto w_1(\xi)$ is a group homomorphism since $w_1(\xi \otimes \eta) = w_1(\xi) + w_1(\eta)$ (7-C in [MS], proven using universal bundle). Consider the universal bundle,

$$
\begin{array}{ccccc}
E(\xi) & \xrightarrow{\sim} & f_\xi^* E(\gamma^1) & \longrightarrow & E(\gamma^1) \\
 & \searrow & \downarrow & & \downarrow \\
 & & B & \xrightarrow{f_\xi} & \mathbb{P}^\infty
\end{array}
$$

we have

$$[B, \mathbb{P}^\infty] \xrightarrow{\psi} Vect_1(B) \xrightarrow{\varphi} H^1(B; \mathbb{Z}/2Z)$$

where $\psi([f]) = f^*(\gamma^1), \varphi(\xi) = w_1(\xi)$, and the composition is

$$[f] \mapsto f^*(\gamma^1) \mapsto w_1(f^*(\gamma^1)) = f^* w_1(\gamma^1).$$

The map ψ is bijective by the 2 properties of universal bundle. The composition is bijective, for $w_1(\gamma^1)$ is a generator of $H^1(\mathbb{P}^\infty, \mathbb{Z}/2\mathbb{Z})$, and the isomorphism $[B, K(G,1)] \cong H^1(B; G)$ for $G = \mathbb{Z}/2\mathbb{Z}$,since $\mathbb{P}^\infty = K(\mathbb{Z}/2\mathbb{Z}, 1)$.

Thus $\psi : Vect_1(B) \cong H^1(B; \mathbb{Z}/2\mathbb{Z})$, the injectivity implies *(2)*.

(4) No, consider $\mathbb{P}^5$ and its tangent bundle, since $w_1(\mathbb{P}^5) = 0, w_3(\mathbb{P}^5) = a^3 \neq 0$, it must be non-trivial.

6. Algebraic topology 2: HW6

Problem 1.(6-B in [MS]) Show that the restriction homomorphism

$$i^* : H^p(G_n(\mathbb{R}^\infty)) \to H^p(G_n(\mathbb{R}^{n+k}))$$

is an isomorphism for $p < k$, any coefficient group may be used.

Solution. For $r \leqslant k$, the number of r-cells in $G_n(\mathbb{R}^{n+k})$ is exactly the number of partitions of r into at most n integers, it remains the same for $G_n(\mathbb{R}^\infty)$. Thus $i : G_n(\mathbb{R}^{n+k}) \to G_n(\mathbb{R}^\infty)$ restricts to an homeomorphism on the k-skeleton, according to cellular cohomology, $i^* : H^p(G_n(\mathbb{R}^\infty)) \to H^p(G_n(\mathbb{R}^{n+k}))$ must be an isomorphism for $p < k$.

Problem 2.(6-C in [MS]) Show that the correspondence $f : X \to \mathbb{R}^1 \oplus X$ defines an embedding of the Grassmann manifold $G_n(\mathbb{R}^m)$ into $G_{n+1}(\mathbb{R}^1 \oplus \mathbb{R}^m) = G_{n+1}(\mathbb{R}^{m+1})$, and that f is covered by a bundle map

$$\varepsilon^1 \oplus \gamma^n(\mathbb{R}^m) \to \gamma^{n+1}(\mathbb{R}^{m+1}).$$

Show that f carries the r-cell of $G_n(\mathbb{R}^m)$ which corresponds to a given partition $i_1 \cdots i_s$ of r onto the r-cell of $G_{n+1}(\mathbb{R}^{m+1})$ which corresponds to the same partition $i_1 \cdots i_s$.

Solution. Consider the Stiefel manifolds

$$\begin{array}{ccc}
(\mathbb{R}^m)^n & \xrightarrow{\widetilde{f}} & (\mathbb{R}^{m+1})^{n+1} \\
\uparrow & & \uparrow \\
V_n(\mathbb{R}^m) & \xrightarrow{\overline{f}} & V_{n+1}(\mathbb{R}^{m+1}) \\
\downarrow & & \downarrow \\
G_n(\mathbb{R}^m) & \xrightarrow{f} & G_{n+1}(\mathbb{R}^{m+1})
\end{array}$$

f naturally gives an embedding $\widetilde{f} : (\mathbb{R}^m)^n \to (\mathbb{R}^{m+1})^{n+1}$, and it restricts to an embedding $\overline{f} : V_n(\mathbb{R}^m) \subset (\mathbb{R}^m)^n \to V_{n+1}(\mathbb{R}^{m+1}) \subset (\mathbb{R}^{m+1})^{n+1}$. f is the induced map of $\overline{f}$ via quotient, so is also an embedding.
Define $F : \varepsilon^1 \oplus \gamma^n(\mathbb{R}^m) \to \gamma^{n+1}(\mathbb{R}^{m+1})$ by

$$(L, (a, u)) \mapsto (\mathbb{R}^1 \oplus L, (a, u)), a \in \mathbb{R}, u \in L \subset \mathbb{R}^m$$

then F gives an isomorphism on each fiber, and from definition, $\pi \circ F = f \circ p$, so the following diagram commute, thus F is a bundle map.

$$\begin{array}{ccc} \varepsilon^1 \oplus \gamma^n(\mathbb{R}^m) & \xrightarrow{F} & \gamma^{n+1}(\mathbb{R}^{m+1}) \\ p\big\downarrow & & \big\downarrow\pi \\ G_n(\mathbb{R}^m) & \xrightarrow{f} & G_{n+1}(\mathbb{R}^{m+1}) \end{array}$$

As for the r-cells, suppose an r-cell of $G_n(\mathbb{R}^m)$ corresponding to a partition $i_1 \cdots i_s$ is given by Schubert symbol $\sigma = (\sigma_1 \cdots \sigma_n)$, with $r = \sigma_1 - 1 + \cdots + \sigma_n - n$, and $\sigma_1 - 1, \cdots \sigma_n - n$ the same as $i_1, \cdots, i_s$ up to a cancellation of zeros. Then f maps the cell to a cell given by Schubert symbol $\sigma' = (\sigma'_1 \cdots \sigma'_{n+1})$, where

$$\sigma'_1 = 1, \sigma'_2 = \sigma_1 + 1, \cdots, \sigma'_{n+1} = \sigma_n + 1.$$

Note $\sigma'_1 - 1 = 0, \sigma'_2 - 2 = \sigma_1 - 1, \cdots, \sigma'_{n+1} - (n+1) = \sigma_n - n$, so it corresponds to the same partition $i_1 \cdots i_s$, and is an r-cell of $G_{n+1}(\mathbb{R}^{m+1})$.

Problem 3.(6-D in [MS]) Show that the number of distinct Stiefel-Whitney numbers for an n-dimensional manifold is equal to $p(n)$.

Solution. The required number is the number of $(r_1 \cdots r_n)$ with $1 \cdot r_1 + \cdots + n \cdot r_n = n$ and $r_i \geqslant 0$. Let $s_1 = r_n, s_2 = r_n + r_{n-1}, \cdots, s_n = r_n + \cdots + r_1$, then $0 \leqslant s_1 \leqslant \cdots \leqslant s_n$ and $s_1 + \cdots + s_n = n$, which means $(s_1 \cdots s_n)$ is a partition of n. Conversely, given a partition $(s_1 \cdots s_n)$, let $r_n = s_1, r_{n-1} = s_2 - s_1, \cdots, r_1 = s_n - s_{n-1}$, then $1 \cdot r_1 + \cdots + n \cdot r_n = n$. Thus the required number equals the number of partitions of n, i.e. $p(n)$.

Problem 4.(7-A in [MS]) Indentify explicitly the cocycle in $C^r(G_n) \cong H^r(G_n)$ which corresponds to the Stiefel-Whitney class $w_r(\gamma^n)$.

Solution. According to **Problem 1.**, we have $C^r(G_n) \cong H^r(G_n) \cong H^r(G_n(\mathbb{R}^{n+r+1}))$. And using **Problem 2.**, we can map the r-cell of $G_1(\mathbb{R}^{r+2}) = \mathbb{P}^{r+1}$ to a r-cell of $G_n(\mathbb{R}^{n+r+1})$. Since the partition of r for $w_r(\mathbb{P}^{r+1})$ should be $(1 \cdots 1)$, (for $(0 \cdots 1)$ with $0 + \cdots + 1 \cdot r = r$ gives partition $(1, \cdots, 1)$ via **Problem 3.**), that is the same as the partition of r corresponding to $w_r(\gamma^n)$. As a result,

$$f : C^r(G_n) \cong H^r(G_n) \cong H^r(G_n(\mathbb{R}^{n+r+1})) \xrightarrow{(\mathbb{R}^{n-1}\oplus)^*} H^r(\mathbb{P}^{r+1})$$

satisfies $f^* w_r(\mathbb{P}^{r+1}) = w_r(G_n)$, In other word, the cocycle in $C^r(G_n)$ is the inverse image of the r-cell of $\mathbb{P}^{r+1}$.

Problem 5. Consider a vector bundle ξ over a paracompact base B. We have $\mathbb{R}^n \to E \xrightarrow{\pi} B$. Let $F(E)$ denote the space of frames in E:

$$F(E) := \{(b, L_1, \cdots, L_n) \mid L_i\text{'s are linearly independent lines in } F_b\}.$$

Let $f : F(E) \to B$ denote the natural projection. Prove that

(1) $f^*\xi$ is isomorphic to a Whitney sum of 1-dimensional sub-bundles.

(2) f induce an injective map on cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

Solution. *(1)* The map $F : F^*E(\xi) \to E(\xi)$ is given by

$$\begin{array}{ccc} f^*E(\xi) & \xrightarrow{F} & E(\xi) \\ p\big\downarrow & & \big\downarrow\pi \\ F(E) & \xrightarrow{f} & B \end{array}$$

$(b, L_1 \cdots, L_n, v) \mapsto (b, v)$, where $v \in F_b$. Now write (uniquely) $v = v_1 + \cdots v_n, v_i \in L_i$, then $p_i : (b, L_1, \cdots, L_n, v_i) \mapsto (b, L_1, \cdots, L_n)$ gives a 1-dimensional sub-bundle η_i of $f^*E(\xi)$ for each i. Also we have $f^*E(\xi) = \eta_1 \oplus \cdots \oplus \eta_n$.

(2) We prove this result by induction for

$$F(E)_k := \{(b, L_1, \cdots, L_k) \mid L_i\text{'s are linearly independent lines}\}$$

where $1 \leqslant k \leqslant n$. For $k = 1$, $\mathbb{P}^n \to F(E)_1 \to B$ is a fiber bundle, and the canonical line bundle L over $F(E)_1$, given by $(b, L, v) \mapsto (b, L), v \in L$, restricts naturally to the canonical line bundle L' over $\mathbb{P}^1$. Note that $L' \in Vect_1(\mathbb{P}^n) \cong H^1(\mathbb{P}^n; \mathbb{Z}/2\mathbb{Z})$ is a generator of $H^*(\mathbb{P}^n; \mathbb{Z}/2\mathbb{Z})$, so $H^*(F(E)_1; Z/2\mathbb{Z}) \twoheadrightarrow H^*(\mathbb{P}^n; \mathbb{Z}/2\mathbb{Z})$.

Thus we use Leray-Hirsch theorem, which tells $H^*(F(E)_1; \mathbb{Z}/2\mathbb{Z})$ is a free module over $H^*(B; \mathbb{Z}/2\mathbb{Z})$, or equivalently, $i_1^* : H^*(B; \mathbb{Z}/2Z) \to H^*(F(E)_1; \mathbb{Z}/2\mathbb{Z})$ is injective.

If the result holds for $1 \leqslant k \leqslant n - 1$, then the map similar to **Problem 2.** gives a fibration $\mathbb{P}^n \to F(E)_{k+1} \to F(E)_k$, so apply the procedure above, we get an injective map $i_{k+1}^* : H^*(F(E)_k; \mathbb{Z}/2\mathbb{Z}) \to H^*(F(E)_{k+1}; \mathbb{Z}/2\mathbb{Z})$.

Finally $i_{k+1}^* \circ \cdots i_1^* : H^*(B; \mathbb{Z}/2\mathbb{Z}) \to H^*(F(E)_{k+1}; \mathbb{Z}/2\mathbb{Z})$ is injective. So by induction, the result holds for $1 \leqslant k \leqslant n$.

Problem 6. Suppose ξ and η are vector bundles over B of dimension m, n. Express $w_1(\xi \otimes \eta)$ and $w_2(\xi \otimes \eta)$ in terms of the Stiefel-Whitney classes of ξ, η, and prove your claims.

Solution. $w_1(\xi \otimes \eta) = nw_1(\xi) + mw_1(\eta), w_2(\xi \otimes \eta) = \frac{n(n-1)}{2}w_1(\xi)^2 + \frac{m(m-1)}{2}w_1(\eta)^2 + nw_2(\xi) + mw_2(\eta) + (mn-1)w_1(\xi)w_1(\eta)$.
The general result is 7-C in [MS], which states

$$w(\xi^m \otimes \eta^n) = p_{m,n}(w_1(\xi), \cdots, w_m(\xi), w_1(\eta), \cdots, w_n(\eta)),$$

where $p_{m,n}(\sigma_m, \cdots, \sigma_m, \sigma'_1, \cdots, \sigma'_n) = \prod_i \prod_j (1 + t_i + t'_j)$.
(For w_2, consider the degree 2 terms in $\prod_i \prod_j (1 + t_i + t'_j)$, which is a sum of $2mn(mn-1)$ monomials, and can be written as $\frac{n(n-1)}{2}(\Sigma t_i)^2 + \frac{m(m-1)}{2}(\Sigma t'_j)^2 + n\Sigma_{i<k} t_i t_k + m\Sigma_{j<l} t'_j t'_l + (mn-1)(\Sigma t_i)(\Sigma t_j)$, by counting occurence)
Now we prove the general result.
First consider the line bundle case, i.e. $m = n = 1$. Let γ be the canonical line bundle of $\mathbb{P}^\infty$, p_1, p_2 projections from $\mathbb{P}^\infty \times \mathbb{P}^\infty \to \mathbb{P}^\infty$, and $\mu : \mathbb{P}^\infty \times \mathbb{P}^\infty \to \mathbb{P}^\infty$ a map

$$\begin{array}{ccc} p_1^*\gamma \otimes p_2^*\gamma & \longrightarrow & \gamma \\ \downarrow & & \downarrow \\ \mathbb{P}^\infty \times \mathbb{P}^\infty & \xrightarrow{\mu} & \mathbb{P}^\infty \end{array}$$

with the diagram commuting. Then we can write $\mu^* w_1(\gamma) = w_1(p_1^*\gamma \otimes p_2^*\gamma) = ap_1^* w_1(\gamma) + bp_2^* w_1(\gamma)$ according to Künneth formula. Note that for a permutation $\sigma \neq \text{id} \in S^2$, $\mu \circ \sigma$ also satisfies the properties of μ, so by the homotopy property of universal bundle, $\mu \circ \sigma \simeq \mu$. As a result,

$$\begin{aligned} ap_2^* w_1(\gamma) + bp_1^* w_1(\gamma) &= (\mu \circ \sigma)^* w_1(\gamma) \\ &= \mu^* w_1(\gamma) = ap_1^* w_1(\gamma) + bp_2^* w_1(\gamma), \end{aligned}$$

which means $a = b \in \mathbb{Z}/2\mathbb{Z}$. Now let $f_i : B \to \mathbb{P}^\infty$ with

$$\begin{array}{ccc} \xi & \longrightarrow & \gamma \\ \downarrow & & \downarrow \\ B & \xrightarrow{f_1} & \mathbb{P}^\infty \end{array} \qquad \begin{array}{ccc} \eta & \longrightarrow & \gamma \\ \downarrow & & \downarrow \\ B & \xrightarrow{f_2} & \mathbb{P}^\infty \end{array}$$

then

$$\begin{aligned} (f_1, f_2)^* \mu^* \gamma &= (f_1, f_2)^* (p_1^*\gamma \otimes p_2^*\gamma) \\ &\cong (p_1 \circ (f_1, f_2))^* \gamma \otimes (p_2 \circ (f_1, f_2))^* \gamma \\ &= f_1^*\gamma \otimes f_2^*\gamma = \xi \otimes \eta \end{aligned}$$

Thus $w_1(L_1 \otimes L_2) = (f_1, f_2)^* \mu^* w_1(\gamma)$. Because in general $w_1(L_1 \otimes L_2) \neq 0$, so $a, b \neq 0$ above, i.e. $a = b = 1$. In return,

$$\begin{aligned} w_1(L_1 \otimes L_2) &= (f_1, f_2)^* \mu^* w_1(\gamma) \\ &= (f_1, f_2)^* (p_1^* w_1(\gamma) + p_2^* w_1(\gamma)) \\ &= (p_1 \circ (f_1, f_2))^* w_1(\gamma) + (p_2 \circ (f_1, f_2))^* w_1(\gamma) \\ &= f_1^* w_1(\gamma) + f_2^* w_1(\gamma) = w_1(\xi) + w_1(\eta). \end{aligned}$$

For general cases, using splitting principle, there is a paracompact space Y and a map $f : Y \to B$, suth that

- $f^*\xi \cong \xi_1' \oplus \cdots \oplus \xi_m'$;
- $f^* : H^*(B; \mathbb{Z}/2\mathbb{Z}) \to H^*(Y; \mathbb{Z}/2\mathbb{Z})$ is injective.

Again, there is a paracompact space X and a map $g : X \to Y$ such that

- $g^* f^* \eta = \eta_1 \oplus \cdots \oplus \eta_n$;
- $g^* : H^*(Y; \mathbb{Z}/2\mathbb{Z}) \to H^*(X; \mathbb{Z}/2\mathbb{Z})$ is injective.

Let $\xi_i = g^* \xi_i'$, then $g^* f^* \xi = \xi_1 \oplus \cdots \oplus \xi_m$, and

$$g^* f^* (\xi \otimes \eta) \cong (\oplus_i \xi) \otimes (\oplus_j \eta_j),$$

as a result

$$\begin{aligned} w(g^* f^* (\xi \otimes \eta)) &= w((\oplus_i \xi_i) \otimes (\oplus_j \eta_j)) \\ &= \prod_i \prod_j w(\xi_i \otimes \eta_j) \\ &= \prod_i \prod_j (1 + w_1(\xi_i) + w_1(\eta_j)). \end{aligned}$$

Write the last term as $q(t_i, t_j') = p_{m,n}(\sigma_i, \sigma_j')$, $t_i = w_1(\xi_i), t_j' = w_1(\eta_j)$, since it remains the same after permutations of t_i or of t_j'. By construction of Stiefel-Whitney class, $\sigma_i = g^* f^* w_i(\xi), \sigma_j' = g^* f^* w_j(\eta)$, so

$$\begin{aligned} g^* f^* w(\xi \otimes \eta) &= p_{m,n}(\sigma_i, \sigma_j') \\ &= p_{m,n}(g^* f^* w_i(\xi), g^* f^* w_j(\eta)) \\ &= g^* f^* p_{m,n}(\sigma_i, \sigma_j'). \end{aligned}$$

But $g^* f^*$ is injective, so $w(\xi \otimes \eta) = p_{m,n}(\sigma_i, \sigma_j')$.

7. Algebraic topology 2: HW7

Problem 1. We can similarly define the Euler class $\overline{e}(\xi) \in H^n(B;\mathbb{Z}/2\mathbb{Z})$ for any vector bundle ξ, regardless if ξ is orientable or not.

(1) Prove that $\overline{e}(\gamma_1^1)$ is nonzero.

(2) Prove that $\overline{e}(\gamma_1)$ is nonzero.

(3) Prove that $\overline{e}(\xi) = w_1(\xi)$ for any line bundle ξ.

Solution. Similar to $\mathbb{Z}$-case, the Euler class $\overline{e}(\xi)$ should be defined via

$$\begin{array}{ccccccc} H^n(B;\mathbb{Z}/2) & \overset{p^*}{\underset{\sim}{\longrightarrow}} & H^n(E;\mathbb{Z}/2) & \overset{i^*}{\longleftarrow} & H^n(E,E_0;\mathbb{Z}/2) & \longrightarrow & H^n(F,F_0;\mathbb{Z}/2) \\ \overline{e} & \longmapsto & u|_E & \longleftarrow\!\shortmid & u & \longmapsto & u_F \neq 0 \end{array}$$

where u is the Thom class, i.e. $H^k(E;\mathbb{Z}/2\mathbb{Z}) \xrightarrow[\sim]{\bullet \smile u} H^{k+n}(E,E_0;\mathbb{Z}/2\mathbb{Z})$.
For *(1)(2)*, where ξ is non-trivial, $\phi : H^1(B;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} H^2(E,E_0;\mathbb{Z}/2)$,

$$x \mapsto p^*x \smile u,$$

gives $\overline{e}(\xi) = \phi^{-1}(u \smile u)$. Here $u \smile u \neq 0$, since it is the image of $u|_E \neq 0$ under $\bullet \smile u$.
For *(3)*, consider

$$\begin{array}{ccc} \xi = f^*\gamma_1 & \longrightarrow & \gamma_1 \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & \mathbb{P}^\infty \end{array}$$

Since $H^1(\mathbb{P}^\infty;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}w_1(\gamma_1)$ and $\overline{e}(\gamma_1) \neq 0$, we must have $\overline{e}(\gamma_1) = w_1(\gamma)$, thus $w_1(\xi) = f^*w_1(\gamma_1) = f^*\overline{e}(\gamma_1) = \overline{e}(\xi)$.

Problem 2. Prove that an oriented rank 2 vector bundle over a paracompact B is trivial if and only if it has a non-vanishing section.

Solution. "$\Rightarrow$" is obvious. As for "$\Leftarrow$", if ξ^2 has a non-vanishing section, then we can write $\xi = \varepsilon_B^1 \oplus \eta^1$. Since ξ is orientable, $w_1(\eta) = w_1(\xi) = 0$, thus η must be a trivial line bundle, as a result, ξ is a trivial bundle.

Problem 3. Consider a homogeneous polynomial $f(x_0, x_1, x_2)$ of degree d. Let $X = \{f = 0\}$. Such an X is called an algebraic curve of degree d. Let H be the subspace of $\mathbb{CP}^2$ defined by $x_2 = 0$.

(1) Assume that f is non-singular and hence X is a smooth manifold with a fundamental class $[X] \in H_2(\mathbb{CP}^2; \mathbb{Z})$, prove that $[X] = d[H]$.

(2) Prove that if two algebraic curves X and Y of degree d and d' intersect transversely, then they intersect at dd' many points.

Solution. *(1)* We show first that X and H intersect at d points. Pluging in $x_2 = 0$, and divide both sides by x_1^d or x_2^d, we transfer $f = 0$ into a polynomial $F = a_d y^d + \cdots + a_0$, where $y = x_1/x_2$ or x_2/x_1 (take a suitable one). Over $\mathbb{C}$, F must have d zeros, these correspond to the intersection of X and H. For X in generic position, it is transverse.

Since $H \cong \mathbb{CP}^1 \subset \mathbb{CP}^2$, $PD[H] \in H^2(\mathbb{CP}^2)$ is a generator. So $[X] * [H] = [X \cap H] = d[pt]$ tells that $[X] = d[H]$.

(2) For non-singular curves, using *(1)*, $[X \cap Y] = [X] * [Y] = dd'[H] * [H] = dd'[pt]$, i.e. there are dd' many points.

Remark. For general algebraic curves, we shall recall the **Bézout theorem**, which states that if f, g have no common factors ($X = \{f = 0\}, Y = \{g = 0\}$), then

$$dd' = \sum_P I(P, X \cap Y).$$

In transverse case, $I(P, X \cap Y) = 1$ for every $P \in X \cap Y$.

This theorem can be proved (algebraically) by considering for large p, the dimension of degree p part $\dim(\mathbb{C}[x_0, x_1, x_2]/(f, g))_p$, which is equivalent to the right hand side, and equals dd' in this case.

Problem 4. Let M be a manifold. Write down the definition for M to be orientable in AT1. Now assume that M is also a smooth manifold. Write down a definition for the tangent bundle of M to be orientable. Check that a smooth manifold M is an orientable manifold if and only if its tangent bundle is an orientable bundle.

Solution. In AT1, M is orientable if there is an oriented atalas $\{U_\alpha\}$, i.e. the transition function $\varphi_{\alpha\beta}$ fixes $H^n(M|p)$ for any $\alpha, \beta, p \in U_\alpha \cap U_\beta$. The tangent bundle is orientable if $\wedge^n TM$ has a non-vanishing section. Now we check the equivalence. Consider the de Rham cohomology, for $x \in U_\alpha$, and $\omega \in \wedge^n T^* M$, we can take f_α which takes value 1 near x and vanishes outside U_α. Then $f_\alpha \omega$ is a generator of $H^n_{dR,c}(U_\alpha) = H^n(M|x)$. Conversely, for compatible generators ω_α of $H^n_{dR,c}(U_\alpha) = H^n(M|x)$, we can take a partition of unity subordinate to $\{U_\alpha\}$ to get a global form

ω. In conclusion, the existence of a section of $\wedge^n TM$ is equivalent to the existence of a global top form ω, which is equivalent to a compatible choice of generators of each $H^n(M|x)$.

Problem 5. The natural inclusion $\mathbb{R}^{n+1} \subset \mathbb{C}^{n+1}$ induces a map $f : \mathbb{RP}^n \to \mathbb{CP}^n$ given by $[x_0 : \cdots : x_n] \mapsto [x_0 : \cdots : x_n]$. Compute the induced map f^* on the cohomology ring with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

Solution. Recall that $H^*(\mathbb{CP}^n; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[\alpha]/(\alpha^{n+1}), \deg\alpha = 2$, and $H^*(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[\beta]/(\beta^{n+1}), \deg\beta = 1$. So the point is to compute $f^*\alpha \in H^2(\mathbb{RP}; \mathbb{Z}/2\mathbb{Z})$.
For $n = 1$, obviously, $f^*\alpha = 0$. For $n \geqslant 2$, take $\alpha \in H^2(\mathbb{CP}^n; \mathbb{Z}/2)$ with $[H] = PD^{-1}(\alpha)$, and $X \subset \mathbb{RP}^n$ with $[X] \in H_2(\mathbb{RP}^n; \mathbb{Z}/2)$.

$$\begin{aligned}\langle f^*\alpha, [X]\rangle = \langle \alpha, [f(X)]\rangle &= \langle PD[H] \smile PD(f(X)), [\mathbb{CP}^n]\rangle \\ &= \langle PD[H \pitchfork f(X)], [\mathbb{CP}^n]\rangle.\end{aligned}$$

Note that $H \pitchfork f(X) \cong f(\mathbb{RP}^1) \simeq pt \subset \mathbb{CP}^n$ (e.g. $H = \{z_{n-2} = 0\}, X = \{x_0 = \cdots = x_{n-3} = 0\}, H \pitchfork f(X) = \{[0 : \cdots : ka : kb] | a, b \in \mathbb{R}\}$), since $\pi_1(\mathbb{CP}^n) = 0$, thus $\langle f^*\alpha, [X]\rangle = 1$. In conclusion, $f^* : \mathbb{Z}/2\mathbb{Z}[\alpha]/(\alpha^{n+1}) \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}[\beta^2]/(\beta^{n+1}) \subset \mathbb{Z}/2\mathbb{Z}[\beta]/(\beta^{n+1})$.

Problem 6. Let $V_k(\mathbb{C}^n)$ denote the Stiefel manifold consisting of sequences of orthonormal vectors $(v_1, \cdots, v_k)$ in $\mathbb{C}^n$.

(1) Find the largest i such that $V_k(\mathbb{C}^n)$ is i-connected.

(2) Compute the first nontrivial homotopy group $\pi_{i+1}(V_k(\mathbb{C}^n))$.

Solution. For $k = n$, $V_n(\mathbb{C}^n) = U(n)$, $\pi_1(U_n) = \mathbb{Z}$. So consider $k < n$.

(1) $V_k(\mathbb{C}^n) \cong SU(n)/SU(n-k)$,

$$\to \pi_m(SU(n-k)) \to \pi_m(SU(n)) \to \pi_m(V_k(\mathbb{C}^n)) \to$$

and $S^{2n-1} \cong SU(n)/SU(n-1)$,

$$\to \pi_m(SU(n-1)) \to \pi_m(SU(n)) \to \pi_m(S^{2n-1}) \to$$

thus for $m < 2(n-1)$, $\pi_m(SU(n-1)) = \pi_m(SU(n))$, and for $m = 2(n-1)$, $\pi_m(SU(n-1)) \to \pi_m(SU(n))$ is surjective. So for $m < 2(n-k)$, $\pi_m(SU(n-k)) \cong \pi_m(SU(n))$, and for $m = 2(n-k)$,

$$\pi_m(SU(n-k)) \twoheadrightarrow \pi_m(SU(n)) \to \pi_m(V_k(\mathbb{C}^n)).$$

In conclusion, $\pi_m(V_k(\mathbb{C}^n)) = 0$ for $m \leqslant 2(n-k)$. The result is $i = 2(n-k)$, since in *(2)* we shall prove $\pi_{2(n-k)+1}(V_k(\mathbb{C}^n)) = \mathbb{Z}$.

(2) Recall that $H^*(SU(n)) = \wedge(a_3, a_5, \cdots, a_{2n-1})$, so

$$\wedge(a_3, a_5, \cdots, a_{2n-1}) = \wedge(a_3, a_5, \cdots, a_{2(n-k)-1}) \otimes H^*(V_k(\mathbb{C}^n)),$$

since there is no monodromy. From this, $H^{2(n-k)+2}(V_k(\mathbb{C}^n)) = 0$, $H^{2(n-k)+1}(V_k(\mathbb{C}^n)) = \mathbb{Z}$, using

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H^{q+1}(X), \mathbb{Z}) \to H_q(X) \to \mathrm{Hom}_{\mathbb{Z}}(H^q(X), \mathbb{Z}) \to 0,$$

We get $H_{2(n-k)+1}(V_k(\mathbb{C}^n)) = \mathbb{Z}$. Now use Hurewicz theorem, we have $\pi_{2(n-k)+1}(V_k(\mathbb{C}^n)) = \mathbb{Z}$.

8. Algebraic topology 2: HW8

Problem 1. Let $\gamma^\vee$ be the dual of the cannonical bundle over $G_2(\mathbb{C}^4)$, $\Sigma_1 := \{W \in G_2(\mathbb{C}^4) \,|\, \mathbb{C}^2 \cap W \neq 0\}$, σ_1 the Poincaré dual of Σ_1.

(1) Prove that $c_1(\wedge^2\gamma^\vee) = \sigma_1$.

(2) Prove that $c_1(\gamma^\vee) = \sigma_1$.

Solution. *(1)* For any linearly independent $f, g \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^4, \mathbb{C})$, $s : W \mapsto f|_W \wedge g|_W = f|_W \otimes g|_W - g|_W \otimes f|_W$ is a section of $\wedge^2\gamma^\vee$. Note that $\ker f \cap \ker g \cong \mathbb{C}^2$, $1 \leqslant \dim W \cap \ker f \leqslant 2$, similar for g.

- if $\dim W \cap \ker f = 2$ or $\dim W \cap \ker g = 2$, then $W \subset \ker f$ or $\dim W \subset \ker g$, either way, $s(W) \equiv 0$;
- if $\dim W \cap \ker f = \dim W \cap \ker g = 1$, take $u \neq 0 \in W \cap \ker f$ and $v \neq 0 \in W \cap \ker g$.
  - if u, v are linearly dependent, or equivalently, $W \cap \ker f \cap \ker g \neq 0$, take $w \perp u$, then $s(W)(k_1 u + l_1 w, k_2 u + l_2 w) = 0 + l_1 l_2 s(W)(w, w) = 0$, i.e. $s(W) \equiv 0$;
  - if u, v are linearly independent, or equivalently $W \cap \ker f \cap \ker g = 0$, then $s(W)(u, v) = 0 - f(v)g(u) \neq 0$.

In conclusion, $s(W) \equiv 0$ if and only if $W \cap \ker f \cap \ker g \neq 0$, thus $Z_s = \{W \,|\, s(w) \equiv 0\} = \{W \,|\, W \cap \ker f \cap \ker g \neq 0\} = \Sigma_1$.
Thus $c_1(\wedge^2\gamma^\vee) = e(\wedge^2\gamma^\vee) = PD([Z_s]) = \sigma_1$.

(2) Using splitting principle, we may assume $\gamma^\vee$ splits into a sum $\xi \oplus \eta$ of line bundles (for more details, see **Problem 2.**)

$$\begin{aligned} 1 + c_1(\wedge^2\gamma^\vee) = c(\wedge^2\gamma^\vee) &= c(\xi \otimes \eta) \\ &= 1 + c_1(\xi) + c_1(\eta) = 1 + c_1(\gamma^\vee). \end{aligned}$$

Thus $c_1(\gamma^\vee) = c_1(\wedge^2\gamma^\vee) = \sigma_1$.

Problem 2. Suppose ω is a 2-dimensional complex vector bundle. Compute the Chern classes of the third symmetric power $\mathrm{Sym}^3\omega$ in terms of Chern classes of ω.

Solution. Using splitting principle, there is a bundle $f : \omega' = \xi^1 \oplus \eta^1 \to \omega$

such that $f^*: H^k(B(\omega);\mathbb{Z}) \to H^k(B(\omega');\mathbb{Z})$ is injective. Then

$$\begin{aligned}
c(\mathrm{Sym}^3\omega') &= c(\mathrm{Sym}^3(\xi \oplus \eta)) \\
&= c(\xi^{\otimes 3}) \cdot c(\xi^{\otimes 2} \otimes \eta) \cdot c(\xi \otimes \eta^{\otimes 2}) \cdot c(\eta^{\otimes 3}) \\
&= (1 + 3c_1(\xi)) \cdot (1 + 2c_1(\xi) + c_1(\eta)) \\
&\qquad \cdot (1 + c_1(\xi) + 2c_1(\eta)) \cdot (1 + 3c_1(\eta)) \\
\text{(with wolframalpha)} &= 45c^2d^2 + 18c^3d + 18cd^3 \\
&\qquad + 6c^3 + 6d^3 + 48c^2d + 48cd^2 \\
&\qquad + 11c^2 + 11d^2 + 32cd + 6c + 6d + 1 \\
&= 45c_2^2 + 18c_2(c_1^2 - 2c_2) + 6c_1(c_1^2 - 3c_2) \\
&\qquad + 48c_1c_2 + 11(c_1^2 - 2c_2) + 32c_2 + 6c_1 + 1 \\
&= (9c_2^2 + 18c_1^2c_2) + (6c_1^3 + 30c_1c_2) \\
&\qquad + (11c_1^2 + 10c_2) + 6c_1 + 1.
\end{aligned}$$

where $c = c_1(\xi), d = c_1(\eta), c_1 = c_1(\omega'), c_2 = c_2(\omega')$. Since f^* is injective, for ω, similarly, we have

$$\begin{aligned}
c(\mathrm{Sym}^3\omega) = (9c_2(\omega)^2 + 18c_1(\omega)^2c_2(\omega)) + (6c_1(\omega)^3 + 30c_1(\omega)c_2(\omega)) \\
+ (11c_1(\omega)^2 + 10c_2(\omega)) + 6c_1(\omega) + 1.
\end{aligned}$$

Problem 3. Consider complex vector bundles over a paracompact B.

(1) Let $Vect_n(B)$ be the set of isomorphism classes of n-dimensional complex vector bundles over B. Prove that $Vect_1(B)$ forms a group under tensor product.

(2) Suppose ω is a 1-dimensional complex vector bundle over B. Prove that ω is trivial if and only if $c_1(\omega)$ is trivial.

(3) Prove that $Vect_1(B) \cong H^2(B;\mathbb{Z})$ as groups.

Solution. *(1)* The group structure

- the product is $\xi \cdot \eta = \xi \otimes_{\mathbb{C}} \eta$, it is associative and commutative, by the associativity and commutativity (up to an isomorphism) of tensor product on each fiber.
- the zero element is $\varepsilon^1_{\mathbb{C}}$, since $\xi \otimes_{\mathbb{C}} \varepsilon^1_{\mathbb{C}} = \xi$ for any ξ.
- the inverse of ξ is $\xi^\vee = \mathrm{Hom}_{\mathbb{C}}(\xi, \varepsilon^1_{\mathbb{C}})$, by the pairing.

(2) "$\Rightarrow$" is obvious. For "$\Leftarrow$", see *(3)*.

(3) The map $\varphi : Vect_1(B) \to H^2(B;\mathbb{Z}), \xi \mapsto c_1(\xi)$ is a group homomorphism, since $c_1(\xi \otimes_{\mathbb{C}} \eta) = c_1(\xi) + c_1(\eta)$. Consider the universal bundle,

$$\begin{array}{ccccc} E(\xi) & \xrightarrow{\sim} & f_\xi^* E(\gamma^1) & \longrightarrow & E(\gamma^1) \\ & \searrow & \downarrow & & \downarrow \\ & & B & \xrightarrow{f_\xi} & \mathbb{CP}^\infty \end{array}$$

we have

$$[B, \mathbb{CP}^\infty] \xrightarrow{\psi} Vect_1(B) \xrightarrow{\varphi} H^2(B;\mathbb{Z})$$

where $\psi([f]) = f^*(\gamma^1), \varphi(\xi) = c_1(\xi)$, and the composition is

$$[f] \mapsto f^*(\gamma^1) \mapsto c_1(f^*(\gamma^1)) = f^* c_1(\gamma^1).$$

The map ψ is bijective by the 2 properties of universal bundle. The composition is bijective, for $c_1(\gamma^1)$ is a generator of $H^2(\mathbb{CP}^\infty;\mathbb{Z})$, and the isomorphism $[B; K(G,2)] \cong H^2(B;G)$ for $G = \mathbb{Z}$, since $\mathbb{CP}^\infty = K(\mathbb{Z}, 2)$.

Thus $\psi : Vect_1(B) \cong H^2(B;\mathbb{Z})$, the injectivity implies *(2)*.

Problem 4. Let $UT(\Sigma_g)$ be the total space of the unit tangent bundle over a closed oriented surface of genus g. Compute the cohomology groups of $UT(\Sigma_g)$.

Solution. Fiber bundle: $S^1 \xrightarrow{i} X = UT\Sigma_g \xrightarrow{\pi} \Sigma_g$. Recall the Gysin sequence, we have the following long exact sequence

$$\cdots \to H^k(\Sigma_g) \to H^k(X) \to H^{k-1}(\Sigma_g) \xrightarrow{\bullet \smile e} H^{k+1}(\Sigma_g) \to \cdots$$

where e is th Euler class of Σ_g. Recall also $H^i(\Sigma_g) = \begin{cases} \mathbb{Z}, i = 0, 2 \\ \mathbb{Z}^{2g}, i = 1 \\ 0, i \geqslant 3 \end{cases}$, so

$$\begin{array}{l} 0 \to \mathbb{Z} \to H^0(X) \to 0 \to \mathbb{Z}^{2g} \to H^1(X) \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad\qquad\qquad \mathbb{Z} \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \bullet\smile e \\ \qquad\qquad\qquad\qquad\qquad\qquad \mathbb{Z} \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad\qquad\quad H^2(X) \to \mathbb{Z}^{2g} \to 0 \to H^3(X) \to \mathbb{Z} \to 0 \end{array}$$

Here that $\bullet \smile e$ amounts to the multiplication by $\chi(\Sigma_g) = 2 - 2g$. Thus $H^i(X) = \begin{cases} \mathbb{Z} & , i = 0, 3 \\ \mathbb{Z}^{2g+1} & , i = 1, 2 \end{cases}$ for $g = 1$, and

$$H^i(X) = \begin{cases} \mathbb{Z} & , i = 0, 3 \\ \mathbb{Z}^{2g} & , i = 1, \\ \mathbb{Z}^{2g} \oplus \mathbb{Z}/(2-2g)\mathbb{Z} & , i = 2 \end{cases}$$

for $g \neq 1$.

9. Algebraic topology 2: HW9

Problem 1. Consider $\xi := (\gamma_n^1)^\perp$ denote the orthogonal complement of the canonical line bundle γ_n^1 over $\mathbb{RP}^n$.

(1) Fix $u_0 \in \mathbb{RP}^n$. Check that the map $s(u) = u_0 - (u_0 \cdot u)u$ defines a section of the vector bundle ξ which is non-zero on the $(n-1)$-skeleton of $\mathbb{RP}^n$.

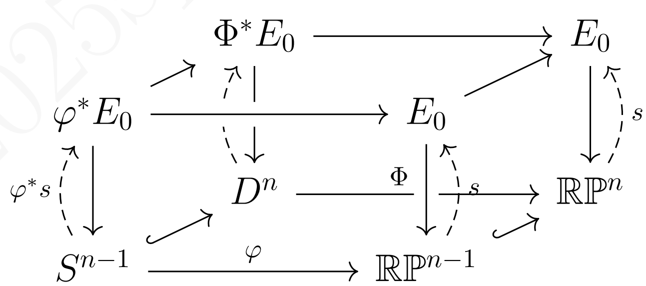
(2) The section s for the bundle $\mathbb{R}^n \backslash \{0\} \to E_0 \to \mathbb{RP}^n$ which is ξ removing zero defines an obstruction cocycle $ob(s) : C_n(\mathbb{RP}^n) \cong \mathbb{Z} \to \pi_{n-1}(\mathbb{R}^n \backslash \{0\}) \cong \mathbb{Z}$. Prove that $ob(s)$ is an isomorphism.

(3) Conclude that the primary obstruction to the bundle $\mathbb{R}^n \backslash \{0\} \to E_0 \to \mathbb{RP}^n$ is non-zero in $H^n(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z})$.

(4) Combining the lecture on Tuesday 4/22 and what you have done in this problem, convince yourself that the first obstruction to a non-zero section of γ^n over G_n modulo 2 is equal to $w_n(\gamma^n)$. Convince yourself that a similar argument works for $w_j(\gamma^n)$ as well.

Solution. *(1)* s is first defined on S^n, and satisfies $s(-u) = s(u)$, also, $u \cdot s(u) = u \cdot u_0 - u \cdot (u_0 \cdot u)u = 0$, so the image lies in ξ. Note that $s(u) = 0$ only for $u = \pm u_0$, so s is non-vanishing on the great circle perpendicular to u_0, which correspond to the $(n-1)$-skeleton.

(2) Consider the following diagram,



which gives $S^{n-1} \to D^n \xrightarrow{\Phi^*s} \Phi^*E_0 \cong D^n \times (\mathbb{R}^n \backslash \{0\}) \to \mathbb{R}^n \backslash \{0\}$. Under this map, n-cell Φ of $\mathbb{RP}^n$ corresponds to $\mathrm{id} : S^{n-1} \to \mathbb{R}^n \backslash \{0\}$, which gives a generator of $\pi_{n-1}(\mathbb{R}^n \backslash \{0\})$. Thus $ob(s)$ is isomorphic. (see the picture on [MS, page 142], the rotation around S^{n-1}, which leaves the vector field invariant, generates $\pi_{n-1}(S^{n-1})$)

(3) As in *(2)*, $ob(s)(\Phi) = [\mathrm{id}] \in \pi_{n-1}(\mathbb{R}^n \backslash \{0\})$. Note that around u_0 the vector field points towards u_0, so the section can not be extended to u_0 continuously. Hence $\overline{ob(s)} \neq 0 \in H^n(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z})$.

(4) (not required)[VBKT, page 104].

Problem 2. Let $F \to E \to B$ be a fiber bundle where B is a CW complex of dimension n. Prove that if F is $(n-1)$-connected, then the bundle always has a section. Prove that if F is $(n-2)$-connected, the bundle has a section if and only if its first obstruction is zero.

Solution. For F being $(k-1)$-connected, from obstruction theory, we can build a section inductively from s_0 on B^0 (which exists trivially), to a section s_k on B^k, and s_k can be extended to s_{k+1} on B^{k+1} if and only if $ob(s_k) = 0$ for the first obstruction class $ob(s_k) \in H^{k+1}(B; \pi_k(F))$.
Take $k = n$ and $B = B^n$, there exists a section on B; take $k = n-1$ and $B = B^n$, if $ob(s_{n-1}) = 0$, then there is a section on B, conversely, using obstruction theory for a section s on B, we must have $ob(s) = 0$.

Problem 3. Use obstruction theory to prove that a smooth compact manifold admits a non-vanishing vector field if $\chi(M) = 0$. This is the converse to the Poincaré-Hopf theorem.

Solution. If M is orientable, from obstruction theory, the first obstruction of the bundle $S^{n-1} \to V_1(TM) \to M$ is the Euler class $e \in H^n(M; \mathbb{Z})$. If $\chi(M) = 0$, then $e = 0$, and we can construct a section on M inductively. Similarly, if M is non-orientable, we can consider the $\mathbb{Z}/2\mathbb{Z}$ Euler class, which is the first obstruction, and is also zero when $\chi(M) = 0$ ([Steenrod, §39]).

Problem 4. Prove that any complex vector bundle over S^1 must be trivial.

Solution. For any $\mathbb{C}^n$-bundle ξ over S^1 , consider the bundle $U(n) = V_n(\mathbb{C}^n) \to V_n(\xi) \to S^1$. The obstruction for a section on $pt \in S^1$ to extend to S^1 is $c_1(S^1) = 0 \in H^2(S^1; \mathbb{Z}) = 0$, so such a section exists. As a result, ξ has n linearly independent sections, i.e. is trivial.

Problem 5. Prove that $\mathrm{Diffeo}^+(S^1)$, the group of orientation-preserving diffeomorphisms of S^1, deformation retracts onto the subgroup $U(1)$.

Solution. Let $\mathrm{Diffeo}_0^+(S^1)$ be the subgroup of $\mathrm{Diffeo}^+(S^1)$ with $0 \in S^1 = \mathbb{R}/\mathbb{Z}$ fixed, then obviously, we have

$$\mathrm{Diffeo}^+(S^1) = \mathrm{Diffeo}_0^+(S^1)U(1).$$

We only have to show that $\mathrm{Diffeo}_0^+(S^1)$ is contractible. For any element $f \in \mathrm{Diffeo}_0^+(S^1)$, it lifts to

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\widetilde{f}} & \mathbb{R} \\ \downarrow & & \downarrow \\ S^1 & \longrightarrow & S^1 \end{array}$$

$\widetilde{f}:\mathbb{R}\to\mathbb{R}$, with $\widetilde{f}(x+1)=\widetilde{f}(x)+d, d\in\mathbb{Z}$, and $\widetilde{f}(0)=0$. Since f is a diffeomorphism, $\widetilde{f}$ should be a homotopy equivalence, so $d=\pm 1$. For f to be orientation-preserving, d must be 1. In this case,

$$H(x,t)=tx+(1-t)\widetilde{f}(x)$$

gives a homotopy between $\widetilde{f}$ and id_{S^1}, thus $\mathrm{Diffeo}_0^+(S^1)$ retracts to id_{S^1}.

Problem 6. Prove that the topological join of an n-connected space and an m-connected space is $(n+m+2)$-connected.

Solution. The topological join of M,N is given by

$$M*N=M\times N\times[0,1]/\sim$$

where $(a,b_1,0)\sim(a,b_2,0),(a_1,b,0)\sim(a_2,b,0)$. We write $A=M*N=\{(1-t)a+tb\,|\,a\in M,b\in N\}$ for simplicity.
There is an isomorphism (see G. Whitehead's paper or J. Milnor's paper)

$$\widetilde{H}_{k+1}(M*N)\cong\sum_{i+j=k}\widetilde{H}_i(M)\otimes\widetilde{H}_j(N)\oplus\sum_{i+j=k-1}\mathrm{Tor}^1_{\mathbb{Z}}(\widetilde{H}_i(M),\widetilde{H}_j(B)).$$

From Hurewicz's theorem, $\pi_{\leqslant m+n+2}(A)=H_{\leqslant m+n+2}(A)=0$, i.e. A is $(m+n+2)$-connected.
Here is a sketch of proof (from Milnor's). There is a MV sequence

$$\to\widetilde{H}_{k+1}(M*N)\to\widetilde{H}_k(\overline{M}\cap\overline{N})\xrightarrow{\psi}\widetilde{H}_k(\overline{M})\oplus\widetilde{H}_k(\overline{N})\xrightarrow{\phi}\widetilde{H}_k(M*N)\to$$

where $\overline{M}=\left\{a\in A\,|\,t\leqslant\frac{1}{2}\right\},\overline{N}=\left\{a\in A\,|\,t\geqslant\frac{1}{2}\right\}$ and thus $\overline{M}\cap\overline{N}\cong M\times N,\overline{M}\simeq M,\overline{N}\simeq N$. Since $i_1:M\to M*N$ and $i_2:N\to M*N$ are null-homotopic (taking $t=1,0$ respectively), ϕ must be trivial.

$$0\to\widetilde{H}_{k+1}(M*N)\to\widetilde{H}_k(M\times N)\xrightarrow{\psi}\widetilde{H}_k(M)\oplus\widetilde{H}_k(N)\to 0$$

Now we get the required formula from the above (splitting) exact sequence and Künneth formula.

10. Algebraic topology 2: HW10

Problem 1. Compute the total Stiefel-Whitney class of the tangent bundle of $\mathbb{CP}^n$.

Solution. Recall that the total Chern class of $\mathbb{CP}^n$ is $(1+a)^{n+1}$, where $a \in H^2(\mathbb{CP}^n;\mathbb{Z})$ is a generator, and also that the coefficient homomorphism

$$H^*(\mathbb{CP}^n;\mathbb{Z}) \to H^*(\mathbb{CP}^n;\mathbb{Z}/2\mathbb{Z})$$

sends $c(\mathbb{CP}^n)$ to $w(\mathbb{CP}^n)$. Thus the total Stiefel-Whitney class is $(1+\alpha)^{n+1}$, where $\alpha \in H^2(\mathbb{CP}^n;\mathbb{Z}/2\mathbb{Z})$ is a generator.

Problem 2. Compute the total Pontrjagin class of the tangent bundles of S^n and $\mathbb{CP}^n$.

Solution. *(1)* For S^n, since $TS^n \oplus \varepsilon^1_{S^n} = \varepsilon^{n+1}_{S^n}$, $p(S^n) = p(\varepsilon^{n+1}_{S^n}) = 0$.

(2) For $\mathbb{CP}^n$, we have

$$\begin{aligned} 1 - p_1 + \cdots + (-1)^n p_n &= c(\mathbb{CP}^n) \cdot c(\overline{\mathbb{CP}^n}) \\ &= (1-a^2)^{n+1} \end{aligned}$$

where $a \in H^2(\mathbb{CP}^n)$ is a generator. Thus the total Pontrjagin class is $(1+a^2)^{n+1}$.

Problem 3. (Stiefel-Whitney v.s. Pontrjagin classes) Prove that

$$w_{2i}^2(\xi) = p_i(\xi) \mod 2$$

in $H^{4i}(B;\mathbb{Z}/2\mathbb{Z})$ for each i.

Solution. The coefficient homomorphism

$$f : H^*(B;\mathbb{Z}) \to H^*(B;\mathbb{Z}/2\mathbb{Z})$$

sends $c(\xi)$ to $w(\xi)$. And since

$$p_i(\xi) = (-1)^i \sum_{j=1}^{i} (-1)^j c_j(\xi) c_{2i-j}(\xi) = c_i^2(\xi) \mod 2$$

so $f(p_i(\xi)) = f(c_i^2(\xi)) = w_i^2(\xi)$.

Problem 4. Prove that if a smooth oriented closed manifold M^{4n} is the boundary of an oriented compact manifold V^{4n+1}, then all Pontrjagin numbers of M are zero.

Solution. Note that $TV|_M = TM \oplus \varepsilon^1_M$, so for any Pontrjagin class $p_k(M)$, we have from product formula that $p_k(M) = i^* p_k(V)$.

$$\to H^i(V) \xrightarrow{i^*} H^i(M) \xrightarrow{\delta} H^{i+1}(V, M)$$

So from the exact-ness above, $\delta p_k(M) = 0$. Thus

$$\begin{aligned} \langle p_{i_1}(M) \cdots p_{i_r}(M), [M] \rangle &= \langle p_{i_1}(M) \cdots p_{i_r}(M), \partial [V] \rangle \\ &= \langle \delta(p_{i_1}(M) \cdots p_{i_r}(M)), [V] \rangle = 0 \end{aligned}$$

which means all the Pontrjagin numbers are zero.

Problem 5. Prove that

$$H^*(G_k(\mathbb{R}^\infty); \mathbb{Q}) = \mathbb{Q}[p_1, \cdots, p_{[k/2]}].$$

Solution. Recall that

$$H^*(\widetilde{G_k}(\mathbb{R}^\infty); \mathbb{Q}) = \begin{cases} \mathbb{Q}[p_1, \cdots, p_{[k/2]}] & , k \text{ odd}, \\ \mathbb{Q}[p_1, \cdots, p_{k/2}, e]/(p_{k/2} - e^2) & , k \text{ even} \end{cases}$$

and that $\widetilde{G_k}(\mathbb{R}^\infty)$ is the oriented two-cover of $G_k(\mathbb{R}^\infty)$. According to [Hatcher, §3.G], the map $H^*(G_k(\mathbb{R}^\infty); \mathbb{Q}) \to H^*(\widetilde{G_k}(\mathbb{R}^\infty); \mathbb{Q})$ induced by covering is injective with image $H^*(\widetilde{G_k}(\mathbb{R}^\infty); \mathbb{Q})^{\mathbb{Z}/2\mathbb{Z}}$. Note that the Euler class depends on the orientation, while p_i does not. So we have $H^*(G_k(\mathbb{R}^\infty); \mathbb{Q}) = H^*(\widetilde{G_k}(\mathbb{R}^\infty); \mathbb{Q})^{\mathbb{Z}/2\mathbb{Z}} = \mathbb{Q}[p_1, \cdots, p_{[k/2]}]$.

11. Algebraic topology 2: HW11

Problem 1. For the following statement below, write down a lifting problem that is equivalent to each of the statement below. For example, a real vector bundle ξ over B is orientable if and only if its classifying map $B \to BO(n)$ lifts to a map $B \to BSO(n)$. Draw a commutative diagram for each lifting problem that you write down.

(1) a real vector bundle ξ is a sum of line bundles iff...

(2) a real vector bundle $\xi^n = \eta^k \oplus \mu^{n-k}$ iff...

(3) a real vector bundle ξ has a non-vanishing section iff...

(4) a real vector bundle ξ has k nowhere dependent sections iff...

(5) a real vector bundle ξ^{2n} has a complex structure iff...

Solution. *(1)* its classifying map $B \to BO(n)$ lifts to $B \to B(O(1)^{\oplus n})$;

(2) its classifying map $B \to BO(n)$ lifts to $B \to B(O(k) \times O(n-k))$;

(3) its classifying map $B \to BO(n)$ lifts to $B \to BO(n-1)$;

(4) its classifying map $B \to BO(n)$ lifts to $B \to BO(n-k))$;

(5) its classifying map $B \to BO(2n)$ lifts to $B \to BU(n)$.

Problem 2. Show that a real 2-dimensional vector bundle ξ has a complex structure if and only if $w_1(\xi) = 0$.

Solution. $w_1(\xi) = 0$ if and only if ξ is orientable, which is equivalent to the condition that $B \to BO(2)$ lifts to $BSO(2) = BU(1)$, which happens if and only if ξ has a complex structure, from *(5)* in **Problem 1.**.

Problem 3. The isomorphism $\mathbb{C}^{n+m} \cong \mathbb{C}^n \oplus \mathbb{C}^m$ induces a map of Lie groups $U_n \times U_m \to U_{n+m}$, which further induces a map of classifying spaces:

$$\phi : B(U_n \times U_m) \to BU_{n+m}.$$

Compute the induced map

$$\phi^* : H^*(BU_{n+m}; \mathbb{Z}) \to H^*(BU_n \times BU_m; \mathbb{Z}).$$

Solution. Recall that $H^*(BU(k); \mathbb{Z}) = \mathbb{Z}[c_1, \cdots, c_k]$, where c_i is the i-th Chern class of the universal bundle. Note that

$$
\begin{array}{ccccc}
\gamma_{\mathbb{C}}^{n} \oplus \gamma_{\mathbb{C}}^{m} & = & \phi^{*}\gamma_{\mathbb{C}}^{n+m} & \longrightarrow & \gamma_{\mathbb{C}}^{n+m} \\
& \searrow & \downarrow & & \downarrow \\
& & BU(n) \times BU(m) & \xrightarrow{\phi} & BU(n+m)
\end{array}
$$

Using the Whitney sum formula,

$$\phi^{*}(c(\gamma_{\mathbb{C}}^{n+m})) = c(\gamma_{\mathbb{C}}^{n} \oplus \gamma_{\mathbb{C}}^{m}) = c(\gamma_{\mathbb{C}}^{n}) \cdot c(\gamma_{\mathbb{C}}^{m}),$$

or equivalently, $\phi^{*}c_k(\gamma_{\mathbb{C}}^{n+m}) = \sum\limits_{i+j=k} c_i(\gamma_{\mathbb{C}}^{n}) \cdot c_j(\gamma_{\mathbb{C}}^{m})$.

Problem 4. Consider the determinant map $\det : U_n \to U_1$, compute the induced map

$$H^{*}(BU_1;\mathbb{Z}) \to H^{*}(BU_n;\mathbb{Z}).$$

Solution. Suppose $\det : U(n) \to U(1)$ induces $\phi : BU(n) \to BU(1)$,

$$
\begin{array}{ccccc}
\wedge^{n}\gamma_{\mathbb{C}}^{n} & = & \phi^{*}\gamma_{\mathbb{C}}^{1} & \longrightarrow & \gamma_{\mathbb{C}}^{1} \\
& \searrow & \downarrow & & \downarrow \\
& & G_n(\mathbb{R}^{\infty}) & \xrightarrow{\phi} & \mathbb{CP}^{\infty}
\end{array}
$$

then $\phi^{*}c_1(\gamma_{\mathbb{C}}^{1}) = c_1(\wedge^{n}\gamma_{\mathbb{C}}^{n})$. Using splitting principle, we consider simply the case $\gamma_{\mathbb{C}}^{n} = \oplus_i \xi_i$, where $c_1(\wedge^{n}\gamma_{\mathbb{C}}^{n}) = c_1(\otimes_i \xi_i) = \sum_i c_1(\xi_i) = c_1(\gamma_{\mathbb{C}}^{n})$. Hence $\phi^{*}c_1(\gamma_{\mathbb{C}}^{1}) = c_1(\gamma_{\mathbb{C}}^{n})$, or equivalently, $\phi^{*} : \mathbb{Z}[c_1(\gamma_{\mathbb{C}}^{1})] \cong \mathbb{Z}[c_1(\gamma_{\mathbb{C}}^{n})]$.

Problem 5. Consider the map $f : U_1 \to U_n$ given by $\lambda \mapsto \lambda I$, describe the induced map

$$Bf^{*} : H^{*}(BU_n;\mathbb{Z}) \to H^{*}(BU_1;\mathbb{Z}).$$

Solution. Suppose $f : U(1) \to U(n)$ induces $\phi : BU(1) \to BU(n)$,

$$
\begin{array}{ccccc}
(\gamma_{\mathbb{C}}^{1})^{\oplus n} & = & \phi^{*}\gamma_{\mathbb{C}}^{n} & \longrightarrow & \gamma_{\mathbb{C}}^{n} \\
& \searrow & \downarrow & & \downarrow \\
& & \mathbb{CP}^{\infty} & \xrightarrow{\phi} & G_n(\mathbb{R}^{\infty})
\end{array}
$$

then $\phi^{*}c(\gamma_{\mathbb{C}}^{n}) = (c(\gamma_{\mathbb{C}}^{1}))^{n} = (1 + c_1(\gamma_{\mathbb{C}}^{n}))^{n}$, or equivalently we have $\phi^{*} : \mathbb{Z}[c_i(\gamma_{\mathbb{C}}^{n})] \cong \mathbb{Z}[\binom{n}{i} c_1(\gamma_{\mathbb{C}}^{1})^{i}]$.

Problem 6. Consider a smooth orientable circle bundle $S^1 \to E \to B$ over a CW complex B. Prove that this bundle is trivial if it has a continuous section.

Solution. "$\Rightarrow$" is obvious. For "$\Leftarrow$", consider the correspondence

$$\begin{Bmatrix} \text{principal bundle} \\ \mathrm{Diff}^+(S^1) \to P \to B \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} \text{orientable } S^1\text{-bundle} \\ F \to E \to B \end{Bmatrix}$$

which is given by $P \to B \mapsto P \times_{\mathrm{Diff}^+(S^1)} S^1 \to B$. Recall in **HW 9.** that $\mathrm{Diff}^+(S^1) \simeq U(1) = S^1$. Thus $E = P \times_{U(1)} S^1 \cong P$, (in general, $P \times_G (G/H) \cong P/H$), which means we can regard $E \to B$ as a principal $U(1)$ bundle up to a homotopy equivalence.
Recall that any principal bundle is trivial if and only if it has a continuous section, thus $E \to B$ is trivial in this case.